Hidden Large Deviations in regularly varying Lévy processes

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ANZAPW 2015

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Regular variation

- \( X \sim F \) has a regularly varying tail (or heavy-tail) with tail index \( \alpha > 0 \) if

\[
\bar{F}(x) := \mathbb{P}(X > x) = x^{-\alpha} \mathcal{L}(x),
\]

where \( \mathcal{L}(tx)/\mathcal{L}(t) \to 1 \) as \( t \to \infty \). Or, for \( x > 0 \),

\[
\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha} := \nu_\alpha(x, \infty).
\]

- Pareto distribution: \( \bar{F}(x) = x^{-\alpha}, x \geq 1 \).
Multivariate Regular Variation (MRV)

Suppose $\{0\} \in C_0 \subset C \subset [0, \infty)^d$. Then $Z \in C$ is regularly varying on $C \setminus C_0$, if there exist a scaling function $b(t) \in RV_{1/\alpha}, \alpha > 0$ and a non-null limit measure $\nu \in M(C \setminus C_0)$ such that as $t \to \infty$

$$t \mathbb{P} \left[ \frac{Z}{b(t)} \in \cdot \right] \to \nu(\cdot) \text{ in } M(C \setminus C_0).$$

Write $Z \in MRV(\alpha, b, \nu, C \setminus C_0)$.

- Think $d = 2$: $C = [0, \infty)^2$, $C_0 = \{0\}$. Then $E = C \setminus C_0$.
- Asymptotic independence $\Rightarrow \nu((0, \infty)^2) = 0$.
- E.g.: $(X, Y)$ has Pareto(1) margins + Gaussian Copula with correlation $\rho < 1$. Then

$$\nu \left( [0, (x, y)]^c \right) = \frac{1}{x} + \frac{1}{y}.$$
Hidden Regular Variation (HRV)

Let \( \mathbb{E} = [0, \infty)^2 \setminus \{0\} \) and \( \mathbb{E}_0 = [0, \infty)^2 \setminus \{\text{axes}\} = (0, \infty)^2 \).

**Definition:** Z is MRV on \( \mathbb{E} \) with HRV on \( \mathbb{E}_0 = (0, \infty)^2 \) if for \( 0 < \alpha \leq \alpha_0 \),

- \( Z \in MRV(\alpha, b, \nu, \mathbb{E}) \) and \( Z \in MRV(\alpha_0, b_0, \nu_0, \mathbb{E}_0) \) with
  - \( \lim_{t \to \infty} \frac{b(t)}{b_0(t)} \to \infty \) as \( t \to \infty \).

Maulik and Resnick [2005], Mitra and Resnick [2011]

*Hidden Large Deviations for regularly varying Lévy processes*
Stable subordinators

1. Suppose \( \{X_t\}_{t \geq 0} \) is a Lévy process.
   - \( X_0 = 0 \) and it has stationary independent increments.
   - It has a version with almost surely right continuous paths with left limits (càdlág).

2. A Levy subordinator is a Levy process with non-decreasing sample paths.

3. A Levy subordinator is called a Stable subordinator if moreover,
   \[
   X_t \overset{d}{=} t^{1/\alpha} X_1, \quad \forall t > 0.
   \]
   Here \( \alpha \in (0, 1) \) and is called the exponent of the stable subordinator.
   - the Lévy measure is given by
   \[
   \nu_\alpha ((x, \infty)) = x^{-\alpha}, \quad x > 0.
   \]
Large deviations of Stable subordinator

- Let $X := \{X_t\}_{t \geq 0}$ be a stable subordinator. Define $X^{(n)} := \{X^n_t\}_{t \in [0,1]}$ as
  \[ X^n_t = X_{nt} \quad n \geq 1. \]

- It is easy to check that for any $\lambda_n \gg n^{1/\alpha}$,
  \[ \frac{X^{(n)}}{\lambda_n} \to 0 \]
  as $\mathbb{D}$-valued random variables under Skorohod metric where $\mathbb{D} := \mathbb{D}([0, 1], \mathbb{R})$.

- Hence for a set $F \subset \mathbb{D}[0, 1]$ bounded away from $0$, then
  \[ Q_n(F) := \mathbb{P} \left[ \frac{X^{(n)}}{\lambda_n} \in F \right] \to 0. \]

- Clearly this is a rare event. At what rate does it happen?
From Hult et al. [2005], we have that there exists a measure $\mu^{(0)}$ on $\mathbb{D} \setminus \{0\}$ such that
\[
\frac{\lambda_n^\alpha}{n} \mathbb{P} \left[ \frac{X^{(n)}}{\lambda_n} \in F \right] \to \mu^{(0)}(F) \tag{1}
\]
for all $F \subset \mathbb{D}$ bounded away from $\mathbb{D}_0 = \{\text{constant functions}\}$ satisfying $\mu^{(0)}(\partial F) = 0$.

Moreover, $\mu^{(0)}$ is supported on
$$\mathbb{D}_{=1} = \{f \in \mathbb{D} : f \text{ has only one jump}\}.$$ 

This result is well-known for regularly varying Lévy processes, random walks with regularly varying (sub-exponential) step-sizes. See Hult et al. [2005].

It supports the idea of one large jump principle for heavy-tails.

Hidden Large Deviations for regularly varying Lévy processes
Hidden large deviations

- Can we find rates of a different kind of rare event?
- What if we look for $X^{(n)}$ belonging to a set $F$ bounded away from $D \leq 1 = D = 0 \cup D = 1$.

According to Hult et al. [2005] we get

$$P \left[ \frac{X^{(n)}}{\lambda_n} \in F \right] = o \left( \frac{n}{\lambda_n^\alpha} \right).$$

- What about the exact rate?
- We restrict to stable subordinators for the talk, but we have shown the result for regularly varying Lévy measures.
We work under the notion of $\mathbb{M}$-convergence; for details see Lindskog et al. [2014], Hult and Lindskog [2006].

We can show that there exists a measure $\mu^{(1)}$ on $\mathbb{D} \setminus \mathbb{D}_{\leq 1}$ such that

$$
\left( \frac{\lambda_n}{n} \right)^2 \mathbb{P} \left[ \frac{X^{(n)}}{\lambda_n} \in F \right] \to \mu^{(1)}(F)
$$

for all $F \subset \mathbb{D}$ bounded away from $\mathbb{D}_{\leq 1} = \{ \text{at most one jump functions} \}$ satisfying $\mu^{(1)}(\partial F) = 0$.

Additionally, $\mu^{(1)}$ is supported on

$$
\mathbb{D}_{=2} = \{ f \in \mathbb{D} : f \text{ has exactly two jumps} \}.
$$

Hence $\mathbb{P} \left[ \frac{X^{(n)}}{\lambda_n} \in F \right] \approx \left( \frac{n}{\lambda_n \alpha_n} \right)^2 \mu^{(1)}(F)$.

This is what we call Hidden Large Deviation at level 1: $\text{HLDP}(1)$. 
Hidden large deviations: HLDP(j)

Proposition: For any \( j \geq 1 \), there exists a measure \( \mu^{(j)} \) supported on \( \mathbb{D}_{j+1} \) such that

\[
\left( \frac{\lambda_n}{n} \right)^{j+1} \mathbb{P} \left[ \frac{X^{(n)}}{\lambda_n} \in F \right] \to \mu^{(j)}(F)
\]

for all \( F \subset \mathbb{D} \) bounded away from \( \mathbb{D}_{\leq j} = \{ \text{at most } j \text{ jump functions} \} \) satisfying \( \mu^{(j)}(\partial F) = 0 \).

Here \( \mathbb{D}_{j+1} = \{ f \in \mathbb{D} : f \text{ has exactly } j + 1 \text{ jumps} \} \).
Idea of Proof: For level 0:

with $N \sim \text{PRM}(\nu_{\alpha} \times \text{Leb})$ where $\nu_{\alpha}(x, \infty) = x^{-\alpha}$, write

$$X_t^n = X_{nt} = a'nt + \int \int_{[0,nt] \times \mathbb{R}_+} x \ N(ds, dx)$$

$$= ant + \int \int_{[0,nt] \times \{ x \leq 1 \}} x \ [N(ds, dx) - ds \ \nu_{\alpha}(dx)] + \int \int_{[0,nt] \times \{ x \leq \frac{1}{n^\alpha} \}} x \ N(ds, dx) + \int \int_{[0,nt] \times \{ x > \frac{1}{n^\alpha} \}} x \ N(ds, dx)$$

We can show that the red part contributes to the limit measure and the rest go to 0.

- Ideas are from Lindskog et al. [2014], Hult et al. [2005].
- We can show this for regularly varying Lévy measures under some conditions on $\lambda_n$. 
References


Das, B. and Roy, P., Hidden large deviations under redulary varying Lévy measures, *under preparation*