Approximating the gains from trade in large markets

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Outline

- Setup market model
- Define the Walrasian quantity traded and level of welfare
- Approximating welfare
- Approximating the scaled quantity traded
- Example computation
Setup

- Consider a market for a homogeneous, indivisible good
- There are $N$ buyers who demand one unit of the good
- There are $M$ sellers with the capacity to produce one unit
- Buyer valuations: $V_1, \ldots, V_N$ are IID RVs with AC DF $F$
- Seller costs: $C_1, \ldots, C_M$ are IID RVs with AC DF $G$
- This is a canonical mechanism design model
Ordering of buyers and sellers

- Sellers are ordered from lowest cost to highest cost

- For $j = 1, \ldots, M$ let $C_{(j)}$ denote the $j$th lowest seller cost

- This forms a supply curve

- Buyers are ordered from highest valuation to lowest valuation

- For $i = 1, \ldots, N$ let $V_i$ denote the $i$th highest buyer valuation

- This forms a demand curve
Ordering of buyers and sellers

Figure 1: The ordered costs and valuations form demand and supply curves
Economic welfare

- The level of welfare generated by a market is essentially the gains from trade.

- The welfare generated by a market is equal to the sum of trading buyer valuations less the sum of trading seller costs.

- If $N^T$ and $M^T$ are the sets of trading buyers and sellers respectively, the level of welfare is given by

$$
\sum_{i \in N^T} V_i - \sum_{j \in M^T} C_j
$$
How can we maximise welfare?

Figure 2: The shaded region shows the maximum level of welfare.
The Walrasian quantity and level of welfare

**Definition**

The *Walrasian quantity*, $K$, is the quantity traded that maximises welfare. That is,

$$K = \arg \max_{k=0,1,\ldots,N \wedge M} \sum_{i=1}^{k} (V[i] - C(i)).$$

The associated level of welfare, $W$, is then

$$W = \sum_{i=1}^{K} (V[i] - C(i)).$$
The problem at hand

We want to approximate

$$W = \sum_{i=1}^{K} (V_{[i]} - C_{(i)})$$

for large $N$ and $M$. 
Background on empirical quantile functions

**Definition**

Let $X_1, \ldots, X_n$ be an IID sample of RVs. Then

$$Q_n(t) = \sum_{i=1}^{n} X_{(i)} \mathbf{1} \left( \frac{i-1}{n} < t \leq \frac{i}{n} \right)$$

is known as the *empirical quantile function* of the sample.
Background on empirical quantile functions

Figure 3: An example of an empirical quantile function
Background on empirical quantile functions

**Proposition (Csörgő and Révész, 1978)**

Let $U_1, \ldots, U_n$ be an IID sample of $U(0, 1)$ RVs and $Q_n(t)$ be the associated uniform quantile function. As $n \to \infty$, for $t \in (0, 1)$,

$$Q_n(t) = t + n^{-\frac{1}{2}} B(t) + \theta_n(t)n^{-1} \log n,$$

where $B(t)$ is a Brownian bridge process and $\|\theta_n\|_\infty = O(1)$ a.s.

Here we use the standard notation $\|\cdot\|_\infty := \sup_{t \in (0, 1)} |\cdot|$. 
Approximating welfare

Returning to the problem at hand,

$$W = \sum_{i=1}^{K} (V[i] - C(i)) = \sum_{i=1}^{K} V[i] - \sum_{i=1}^{K} C(i) \equiv W^V - W^C.$$ 

Approximate $W^V$ and $W^C$ separately.
Approximating $W^C$

- Let $G^{(-1)}$ be the quantile function of $G$

- Let $U_1^C, \ldots, U_M^C$ be an IID sample of $U(0, 1)$ RVs

- Let $Q_M^C$ denote the associated uniform quantile function

- We have

$$W^C = \sum_{i=1}^K C[i] = \sum_{i=1}^K G^{(-1)}\left(U^C_{(i)}\right) = M \sum_{i=1}^K \frac{1}{M} G^{(-1)}\left(Q_M^C\left(\frac{i}{M}\right)\right)$$
Approximating $W^C$

- Rewriting the sum on the previous slide as an integral,

$$W^C = M \sum_{i=1}^{K} \frac{1}{M} G^{(-1)} \left( Q_M^C \left( \frac{i}{M} \right) \right) = M \int_0^{\frac{K}{M}} G^{(-1)} \left( Q_M^C(u) \right) du$$

- Then letting $\mathbb{B}^C$ denote a Brownian bridge process,

$$W^C = M \int_0^{\frac{K}{M}} G^{(-1)} \left( u + M^{-\frac{1}{2}} \mathbb{B}^C(u) + \theta_M^C(u) M^{-1} \log M \right) du,$$

where $\|\theta_M^C\|_\infty = O(1) \text{ a.s.}$
Approximating $W^V$

- Repeating this procedure for $W^V$ we obtain

$$W^V = N \int_0^{K/N} F^{(-1)} \left( 1 - u + N^{-\frac{1}{2}} B^V (1 - u) + \theta_N^V(u) N^{-1} \log N \right) \, du,$$

where $\|\theta_N^V\|_\infty = O(1) \ a.s.$

- Here, $F^{(-1)}$ is the quantile function of $F$.

- $B^V$ is a Brownian bridge process independent of $B^C$.

To proceed, we must approximate $K/N$ and $K/M$. 
Some simulations

Figure 4: Uniformly distributed costs and valuations with $N = 100$ and $M = 110$
Some simulations

Figure 5: Uniformly distributed costs and valuations with $N = 200$ and $M = 220$
Some simulations

Figure 6: Uniformly distributed costs and valuations with $N = 500$ and $M = 550$
Some simulations

Figure 7: Uniformly distributed costs and valuations with $N = 2000$ and $M = 2200$
Approximating \( K/N \)

- Take \( M = \lambda N \) and approximate \( K/N \) only

- Consider the functional

\[
H(k) = \sup \{ t : k(t) < 0 \}
\]

- Let \( G_M^{(-1)}(t) \) and \( F_N^{(-1)}(t) \) denote the empirical quantile functions associated with the seller costs and buyer valuations respectively

- Then

\[
\frac{K}{N} = H \left( G_M^{(-1)}(\lambda^{-1} t) - F_N^{(-1)}(1 - t) \right)
\]
Approximating $K/N$

- For $t \in (0, 1 \wedge \lambda)$ define

$$k_0(t) = G^{(-1)}(\lambda^{-1}t) - F^{(-1)}(1 - t)$$

- Let $t_0 = H(k_0)$ so that, as $n \to \infty$,

$$\frac{K}{N} \xrightarrow{d} t_0$$

- We wish to determine the distribution of the leading order error term
Figure 8: A simulation of $K/N$ with uniform types, $N = 50$ and $M = 50$
Asymptotic normality of sample quantiles

- It is well known that, as $n \to \infty$,

\[
\sqrt{N}(G_M^{-1}(\lambda^{-1}t) - G^{-1}(\lambda^{-1}t)) \xrightarrow{d} \frac{\mathcal{B}^G(\lambda^{-1}t)}{\sqrt{\lambda g(G^{-1}(\lambda^{-1}t))}},
\]

\[
\sqrt{N}(F_N^{-1}(1 - t) - F^{-1}(1 - t)) \xrightarrow{d} \frac{-\mathcal{B}^F(t)}{f(F^{-1}(1 - t))}.
\]

- Combining these gives, as $n \to \infty$,

\[
\sqrt{N}[(G_M^{-1}(\lambda^{-1}t) - F_N^{-1}(1 - t)) - (G^{-1}(\lambda^{-1}t) - F^{-1}(1 - t))] \xrightarrow{d} \frac{\mathcal{B}^G(\lambda^{-1}t)}{\sqrt{\lambda g(G^{-1}(\lambda^{-1}t))}} + \frac{\mathcal{B}^F(t)}{f(F^{-1}(1 - t))}.
\]

- But we want the convergence of

\[
\sqrt{N}[H(G_M^{-1}(\lambda^{-1}t) - F_N^{-1}(1 - t)) - H(G^{-1}(\lambda^{-1}t) - F^{-1}(1 - t))].
\]
The delta method

We need a functional version of the delta method.

Proposition

Let $\theta$, $\sigma^2$ be finite valued constants and suppose $\{X_n\}_{n \in \mathbb{N}}$ satisfies

$$\sqrt{n} \left[ X_n - \theta \right] \overset{d}{\to} \mathcal{N}(0, \sigma^2).$$

Then for any function $r$ with $r'(\theta) \neq 0$,

$$\sqrt{n} \left[ r(X_n) - r(\theta) \right] \overset{d}{\to} \mathcal{N}(0, \sigma [r'(\theta)]^2).$$
The derivative of $H$

- Compute the functional derivative of $H$

- Do this by modifying an example in Section 8, Chapter I of Borovkov (1998)

- For any $v \in C(-\infty, \infty)$,

$$H'(k_0, v) = \frac{v(t_0) \lambda g(G^{-1}(\lambda^{-1}t_0))f(F^{-1}(1 - t_0))}{\lambda g(G^{-1}(\lambda^{-1}t_0)) + f(F^{-1}(1 - t_0))}$$
The Functional Delta Method

By Proposition 1 in Borovkov (1985),

\[
\sqrt{N} \left[ H(G_N^{(-1)}(t) - F_N^{(-1)}(1 - t)) - H(G^{(-1)}(t) - F^{(-1)}(1 - t)) \right] \\
\xrightarrow{d} \frac{\lambda g(G^{(-1)}(\lambda^{-1}t_0)) f(F^{(-1)}(1-t_0))}{\lambda g(G^{(-1)}(\lambda^{-1}t_0)) + f(F^{(-1)}(1-t_0))} \mathcal{N} \left( 0, \frac{t_0(1-\lambda^{-1}t_0)}{\lambda^2 g^2(G^{(-1)}(\lambda^{-1}t_0)) + f^2(F^{(-1)}(1-t_0))} \right) 
\]

\[ \approx Z. \]

Thus, we have

\[
\frac{K}{N} \xrightarrow{d} t_0 + N^{-\frac{1}{2}} Z + O \left( N^{-1} \log N \right),
\]

where \( Z \) is normally distributed.
Uniform costs and valuations with $N = M$

- In this case, our expression for $K/N$ simplifies to

$$
\frac{K}{N} \overset{d}{=} \frac{1}{2} + N^{-\frac{1}{2}} \mathcal{N} \left(0, \frac{1}{8}\right) + O \left(N^{-1} \log N\right) \overset{d}{=} \frac{1}{2} + \frac{Y}{\sqrt{N}} + O \left(N^{-1} \log N\right)
$$

- Our expression for $W^C$ simplifies to

$$
W^C \overset{d}{=} N \int_0^{\frac{1}{2} + N^{-\frac{1}{2}} Y} u \, du + \sqrt{N} \int_0^{\frac{1}{2}} B^C(u) \, du + O \left(\log N\right)
$$

- Our expression for $W^V$ simplifies to

$$
W^V \overset{d}{=} N \int_0^{\frac{1}{2} + N^{-\frac{1}{2}} Y} (1 - u) \, du + \sqrt{N} \int_0^{\frac{1}{2}} B^V (1 - u) \, du + O \left(\log N\right)
$$
Uniform costs and valuations with $N = M$

- Direct computation then gives

\[
W = W^V - W^C = \frac{N}{4} + \sqrt{N}N \left( 0, \frac{10}{192} \right) + O(\log N)
\]

- The dependence of $K$ on $\mathbb{B}^V$ and $\mathbb{B}^C$ affects higher order terms in the expansion
Example error simulation

Figure 9: The distribution of \((W - N/4)/\sqrt{N}\) vs. the density of 
\[ Y \sim \mathcal{N}(0, 10/192) \] for \(N = M = 1000\) with 5000000 simulations.
