Kuramoto Model with Lévy Noise

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An applied problem: the OODA loop

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Kuramoto model of coupled oscillators

The (classic) Kuramoto model is given by the coupled system of ODEs

\[
\frac{d\theta_j^\omega}{dt}(t) = \omega_j - \frac{K}{N} \sum_{i=1}^{N} A_{ij} \sin(\theta_j^\xi(t) - \theta_i^\xi(t))
\]  

(KM)

for \( j = 1, 2, \ldots, N \) where:

- \( K \) is a real parameter,
- \( \{\omega_i\}_{i=1,\ldots,N} \) are natural rotation frequencies,
- the interaction topology is modelled by a graph \( G = (V, E) \) with \( A = [A_{ij}] \) the adjacency matrix and \( N = |V| \).

Classically, \( G \) is a complete graph, i.e., \( A_{ij} = 1 \) for \( i \neq j \) and 0 otherwise.

Hundreds of applications ranging from biological synchronization and rhythmic phenomena, to engineering, etc.
**Synchronisation**

**Definition**

A solution $\theta : \mathbb{R}^+ \rightarrow \mathbb{T}^n$ to the coupled oscillator model (KM) achieves *phase synchronisation* if all phases $\theta_i(t)$ become identical as $t \rightarrow \infty$.

Kuramoto (1975, 1984) showed that *synchronisation* occurs in (KM) for the complete graph if the coupling gain $K$ exceeds a certain threshold $K_{\text{critical}}$ function of the distribution of the natural frequencies $\omega_i$. 
Order parameter

The *order parameter* $r$ given by

$$r(t) = \frac{1}{N} \left| \sum_{i=1}^{N} e^{i \theta_i(t)} \right|$$

is a standard metric for synchronisation.
Network topologies

BA easier to synchronise than ER

Figure: Global synchronisation as graph varies from BA ($\alpha = 0$) to ER ($\alpha = 1$) for $N = 10,000$. $\lambda \approx K/N$. Figure from Gómez-Gardeñes et al. (2007).

$$K_{\text{critical}}(\text{BA}) < K_{\text{critical}}(\text{ER}) < K_{\text{critical}}(\text{SW})$$
Kuramoto model with Gaussian noise

Gaussian noise can be added to the Kuramoto model, giving the coupled system of SDEs

\[ d\theta_j^\xi(t) = \xi_j dt - \frac{K}{N} \sum_{i=1}^{N} A_{ij} \sin(\theta_j^\xi(t) - \theta_i^\xi(t))dt + \sigma dW_j(t) \] (KMG)

for \( j = 1, \ldots, N \) where

- \( \{W_j(\cdot)\}_{j=1,...,N} \) is a family of independent standard Brownian motions that models the thermal noise,

- \( \xi = \{\xi_j\}_{j=1,...,N} \) is a family of IID random variables that models the disorder.

The stochastic evolution is considered once a realisation of the disorder variables \( \xi \) is chosen: the disorder is of quenched type.
Path in Gaussian case

\[ \theta_i(t) \]

\[ r(t) \]

\[ 0 \leq r(t) \leq 1 \]

\[ 0 \leq t \leq 30 \]
**Gaussian noise: Robustness and stochastic synchronisation**

**Figure**: Comparison of $r_\infty$ for increasing $\sigma$ in (KMG). Left figure from Khobasht et al. (2008) and right figure from Esfahani et al. (2012).

*Stochastic synchronisation* can be observed: noise can sometimes help synchronisation.
Questions

What happens when the perturbing noise is heavier-tailed?

\[ K_{\text{critical}}(BA) < K_{\text{critical}}(ER) ? \]
Lévy process

Let $L = (L_t)_{t \geq 0}$ be a Lévy process with canonical triplet $(\gamma, \sigma^2, \Pi)$. Thus the characteristic function of $L$ is given by $E e^{i\theta L_t} = e^{t\Psi(\theta)}$, where

$$\Psi(\theta) = i\theta \gamma - \frac{1}{2} \sigma^2 \theta^2 + \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta x 1_{\{|x|<1\}}) \Pi(dx), \text{ for } \theta \in \mathbb{R}.$$  

We look at two explicit cases:

- **Stable process** $(0, 0, \Pi)$ with: (Subexponential Class)
  
  $$\Pi(dx) = \frac{1}{x^{1+\alpha}} dx, \quad x > 0.$$  

- **Tempered stable** $(\gamma, 0, \Pi)$ with: (Convolution Equivalent Class)
  
  $$\Pi(dx) = \frac{e^{-\lambda x}}{x^{1+\alpha}} dx, \quad x > 0,$$
  
  and $\gamma = (1 - \alpha)^{-1} - \int_0^1 (1 - e^{-\lambda x}) \frac{dx}{x^\alpha}$. 
Lévy noise can be added to the Kuramoto model, giving the coupled system of SDEs

\[ d\theta_j^\xi(t) = \xi_j dt - \frac{K}{N} \sum_{i=1}^{N} A_{ij} \sin(\theta_j^\xi(t) - \theta_i^\xi(t)) dt + \sigma dL_j(t) \]  

for \( j = 1, \ldots, N \) where

- \( \{L_j(\cdot)\}_{j=1,\ldots,N} \) is a family of independent Lévy processes.
- We can vary parameters \( \sigma, \alpha, \) and \( \lambda \)
Simulation

Look at graphs of size $N = 1000$, 1000 paths up to time $T = 60.0$ with a fine time step (5000 steps), average over:

- Initial conditions $\theta_0 \sim \text{Uniform}(\mathbb{T})$
- Disorder (natural frequency) $\xi = 0$
- Graph $G$ (sample of size 30 from each class: ER and BA)

Simulation of:

- Stable $0 < \alpha < 1$ and $1 < \alpha < 2$ (exact)
- Tempered stable $0 < \alpha < 1$ (exact)
- Tempered stable $1 < \alpha < 2$ (approximate depending on param $c$)

See survey by Kawai and Masuda

Parallelised over 1000 cpus, C++ and MPI.
Numerical results

Figure: $\tilde{r}_\infty$ for BA (pink) vs. ER (cyan) as $\alpha$ and $\sigma$ varies for $\lambda = 0.001$. 
Numerical results

Figure: Synchronisation for BA (blue) vs ER (red) as $\sigma$ increases.
Analytic results

Analytic results are very hard (impossible?) to obtain for random $G$ and adding driving noise as well.

- Linearise around a stable point to get tail probabilities
- Estimates for average exit time from a stable point in special cases of graphs in the small noise limit (Freidlin-Wentzell type result)
- If the system escapes a stable point (in finite time), how does it do it? Asymptotically as “well depth” $u \to \infty$. 
How does it exit?

Linearised around a stable point, we can try to understand how the noise pushes the system out of synchronisation.

Since $L_0 = 0$, and conditional on $\tau(u) < T$ we have $L_t > u$ for some $t < T$, in order to get a limit it is natural to scale $L$ by a factor of $u$. Thus, setting

$$L^{(u)}_t = \frac{L_t}{u}, \quad 0 \leq t \leq T,$$  \hfill (2)

we will investigate the limiting behavior of

$$\mathcal{L}(L^{(u)}|\tau(u) < T) \text{ as } u \to \infty,$$  \hfill (3)

where $\mathcal{L}$ denotes the law of the process.
How does it exit? Roughly linearly...

Suppose $\Pi^+(dx)$ is absolutely continuous for sufficiently large $x$ and

$$\Pi^+(dx)/dx \sim \beta x^{\alpha-1} e^{-\lambda x} \text{ as } x \to \infty$$

for some $\beta > 0$, $\alpha > 0$, $r \in \mathbb{R}$.

**Theorem (Griffin and Roberts (2014))**

Let $R$ be the degenerate process defined by

$$R(t) = tT^{-1}, \quad 0 \leq t \leq T,$$

and let $\| \cdot \|_\infty$ denote the sup norm on $D[0, T]$. If $\alpha > 0$, then for any $\delta > 0$,

$$P(\|L(u) - R\|_\infty > \delta \mid \tau(u) < T) \to 0.$$  \hfill (4)

With heavier tails, you see a sudden large jump away from stable point...
AMSI Internship

AMSI Internship with DSTO
FY15/16

PhD student who has just submitted?

Australian citizen
Thank you
Subexponential and Convolution Equivalent

A distribution $F$ on $\mathbb{R}$ with tail $\overline{F} = 1 - F$ belongs to the class $\mathcal{L}(\alpha)$, $\alpha \geq 0$, if
\[
\lim_{u \to \infty} \frac{\overline{F}(u + x)}{\overline{F}(u)} = e^{-\alpha x}, \text{ for } x \in (-\infty, \infty).
\]

Further, $F$ belongs to the class $S(\alpha)$, $\alpha \geq 0$, if in addition
\[
\lim_{u \to \infty} \frac{\overline{F^2}(u)}{\overline{F}(u)} \text{ exists and is finite,} \quad (5)
\]
where $F^2 = F \ast F$. When $F \in S(\alpha)$,
\[
\delta^F_\alpha := \int_{\mathbb{R}} e^{\alpha x} F(x) < \infty, \quad (6)
\]
and the limit in (5) is given by $2\delta^F_\alpha$. Distributions in $S(0)$ are called subexponential, and those in $S(\alpha)$ with $\alpha > 0$, are called convolution equivalent of index $\alpha$. 
For any Lévy measure $\Pi$, let $\Pi^+(\cdot) = \Pi(\cdot \cap (0, \infty))$ and $\overline{\Pi}^+(u) = \Pi^+((u, \infty))$. Then we say

$$\Pi \in S^{(\alpha)} \text{ iff } F \in S^{(\alpha)} \text{ where } \overline{F}(u) = \frac{\overline{\Pi}^+(u)}{\overline{\Pi}^+(1)} \wedge 1, \quad \alpha \geq 0.$$  \hfill (7)

Equivalently, $\overline{\Pi}^+(1)$ may be replaced with $\overline{\Pi}^+(a)$ for any $a > 0$ by closure of $S^{(\alpha)}$ under tail equivalence. Thus $\Pi \in S^{(\alpha)}$ depends only on the positive tail of $\Pi$, and so to emphasize this we will write $\overline{\Pi}^+ \in S^{(\alpha)}$ instead of $\Pi \in S^{(\alpha)}$. Condition (7) is also equivalent to $X_t \in S^{(\alpha)}$ for all (some) $t > 0$. 