Functional Calculus for Phase-type distributions

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Phase–type distribution

Let \( \{X_t\}_{t \geq 0} \) be a Markov jump process with the following properties:
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Phase–type distribution

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**Definition**

*The time until absorption*

\[
\tau = \inf\{t \geq 0 \mid X_t = p + 1\}
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is said to have a phase-type distribution.
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Definition

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- \((\pi, T)\) is a representation of the phase-type distribution.

We write \(\tau \sim \text{PH} (\pi, T)\).
Path of a phase–type distribution
Path of a phase–type distribution

\[ X_t \]

\[ p + 1 \]

\[ p \]

\[ 5 \]

\[ 4 \]

\[ 3 \]

\[ 2 \]

\[ 1 \]

\[ t \]
Path of a phase–type distribution
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Path of a phase-type distribution
If $\tau \sim \text{PH} (\pi, T)$, the density $f$ of $\tau$ is

$$f(x) = \pi \exp(Tx) t.$$
Some properties of Phase–type distributions

- If $\tau \sim \text{PH}(\pi, T)$, the density $f$ of $\tau$ is
  \[ f(x) = \pi \exp(Tx) t. \]

- $T$ is invertible and
  \[ \int_{0}^{\infty} \exp(Tx) \, dx = (-T)^{-1}. \]
Motivation

Let $\tau \sim \text{PH}(\pi, T)$.

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\[ \cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \]

\[
\mathbb{E}(\cos(\tau)) = \int_0^\infty \frac{e^{ix} + e^{-ix}}{2} \pi e^{Tx} t \, dx
\]

\[
= \frac{\pi}{2} \int_0^\infty e^{(i+T)x} dx \, t + \frac{\pi}{2} \int_0^\infty e^{(T-i)x} dx \, t
\]

\[
= \frac{\pi}{2} (-iI - T)^{-1} t + \frac{\pi}{2} (iI - T)^{-1} t
\]

\[
= \pi (-T) (I + T)^{-2} t.
\]
Motivation

What about other functions?
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\[ E(sin(\tau)) = ?. \]
What about other functions?

\[
\mathbb{E}(\sin(\tau)) = ?. \\
\mathbb{E}(\tau^\alpha) = ?, \text{ with } \alpha > 0.
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\[ \mathbb{E}(\sin(\tau)) = ?. \]

\[ \mathbb{E}(\tau^\alpha) = ?, \quad \text{with} \quad \alpha > 0. \]

\[ \mathbb{E}(\omega(\tau)) = ?, \quad \text{with} \quad \omega = ?. \]
Let \( \tau \sim PH(\pi, T) \) and \( \omega \) be a function with Laplace transform

\[
L_\omega(s) = \int_0^\infty e^{-sx} \omega(x) \, dx
\]

which exists for all \( s > 0 \). Then

\[
\mathbb{E}(\omega(\tau)) = \pi L_\omega(-T)t,
\]

where \( t = -Te \).
To prove this theorem observe that:
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Here we will use functional calculus!
Definition (Cauchy integral formula)

Let $A$ be a finite dimensional matrix and let $\gamma$ be a simple closed path which encloses the eigenvalues of $A$. If $f$ is a function which is analytic on and inside the path $\gamma$, then we define

$$f(A) = \frac{1}{2\pi i} \oint_{\gamma} f(z)(zI - A)^{-1} dz.$$
Proof:

We take

\[ \int_0^\infty e^{-sx}\omega(x)\,dx = L_\omega(s) = f(s), \]

because we want to calculate

\[ L_\omega(-T). \]
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- \(-T\) has a finite number of eigenvalues which have positive real part.
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- The Laplace transform \( L_\omega(s) \) is analytic in the positive half-plane.
- \(-T\) has a finite number of eigenvalues which have positive real part.
- We can find a simple closed path \( \gamma \) that encloses all the eigenvalues and this path is located within the positive half-plane.
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- We can find a simple closed path \( \gamma \) that encloses all the eigenvalues and this path is located within the positive half-plane.

Thus we can apply the definition based on the Cauchy formula to calculate \( L_\omega(-T) \).
Calculate $\mathbb{E}(\cos(\tau))$

We have

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$$\mathbb{E}(\cos(\tau)) = \pi ( -T ) ( I + T^2 )^{-1} t.$$
The Mellin transform: $\mathbb{E}(\tau^{\alpha-1})$, $\alpha > 0$.

\[
\mathbb{E}(\tau^{\alpha-1}) = \int_0^\infty x^{\alpha-1} \pi e^{Tx} dt dx = \pi \int_0^\infty x^{\alpha-1} e^{T x} dx dt.
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In this case we have

$$\int_0^\infty x^{\alpha-1} e^{-sx} \, dx = L_{x^{\alpha-1}}(s) = \frac{\Gamma(\alpha)}{s^\alpha}.$$ 

exists for all $\alpha > 0$ and all $s > 0$. 
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The Mellin transform: $E(\tau^{\alpha-1})$, $\alpha > 0$.

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So applying the theorem with $\omega(x) = x^{\alpha-1}$ we get

$$E(\tau^{\alpha-1}) = \pi L_{x^{\alpha-1}}(-T)t = \Gamma(\alpha)\pi(-T)^{-\alpha}t.$$
Objective:

Moment distribution of a phase-type distribution
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Use the functional calculus method to obtain a representation of the Moment distribution of a phase-type distributed random variable.
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What’s the Moment distribution of a random variable?
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Use the functional calculus method to obtain a representation of the Moment distribution of a phase–type distributed random variable.

What’s the Moment distribution of a random variable?

Let $X$ be a non-negative random variable with density $f_X$, then the density given by

$$g_n(x) = \frac{x^n f_X(x)}{\mathbb{E}(X^n)}$$

is called the density of the $n$’th order Moment distribution of $X$. 
In particular, for $\tau \sim \text{PH}(\pi, T)$, we have

$$g_n(x) = \frac{x^n \pi e^{Tx} t}{\mathbb{E}(\tau^n)}.$$
Definition (Matrix–exponential distribution)

A distribution of a non–negative random variable is called matrix–exponential if it has a density function \( f \) on the form

\[
f(x) = \alpha \exp(Sx) \, s,
\]

where \( \alpha \) and \( s \) are row and column vectors respectively and \( S \) is a matrix.
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\((\alpha, S, s)\)

is a representation of the matrix–exponential distribution.
Matrix–exponential distributions

**Theorem**

Consider a phase–type distribution with representation $(\pi, T)$. Then its $n$'th order moment distribution has a matrix–exponential representation given by $(\alpha_n, S_n, s_n)$, where

\[
\alpha_n = \frac{\pi T^{-n}}{\pi T^{-n} e}, 0, \ldots, 0, \quad S_n = \begin{pmatrix}
T & -T & 0 & \cdots & 0 \\
0 & T & -T & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & T \\
\end{pmatrix},
\quad s_n = \begin{pmatrix}
0 \\
0 \\
\vdots \\
t \\
\end{pmatrix}.
\]
Proof:

The density of the Erlang distribution with parameters \((n, \lambda)\) is given by

\[ E_r^n(x; \lambda) = \lambda^n (n-1)! x^{n-1} e^{-\lambda x}, \]

and this corresponds to the \(n-1\) order Moment distribution of an exponential distribution with parameter \(\lambda\).

We can also write this density as follows

\[ E_r^n(x; \lambda) = (1, 0, \ldots, 0) \exp \left( \begin{pmatrix} -\lambda & \lambda \\ \vdots & \ddots & \ddots \\ 0 & \cdots & -\lambda \end{pmatrix} x \right). \]
Proof:

- The density of the **Erlang distribution with parameters** $(n, \lambda)$ is given by

  $$
  f_{\text{Erlang}}(x; n, \lambda) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!},
  $$

  and this corresponds to the $(n-1)$-order moment distribution of an exponential distribution with parameter $\lambda$.
Proof:

The density of the **Erlang distribution with parameters** \((n, \lambda)\) is given by

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\text{Er}_n(x; \lambda) = \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x},
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\[
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\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -\lambda
\end{pmatrix} x \\
0
\end{pmatrix} \begin{pmatrix}
0 \\
0 \\
\vdots \\
\lambda
\end{pmatrix},
\]
The representation of a Moment distribution

The function

\[ z \rightarrow Er_n(x; z) \]

is an analytic function in the positive half–plane,
The representation of a Moment distribution

The function

\[ z \mapsto E_{r_n}(x; z) \]

is an analytic function in the positive half-plane, so the evaluation given by

\[ E_{r_n}(x; -T) \]

is well defined by the Cauchy integral formula.
The representation of a Moment distribution

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is an analytic function in the positive half–plane, so the evaluation given by

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is well defined by the Cauchy integral formula. Then

\[ g_n(x) = \frac{x^n \pi e^{Tx} t}{E(\tau^n)} = \frac{\pi}{E(\tau^n)} (-T)^{-n-1}(n)! \text{Er}_{n+1}(x; -T)t \]
The representation of a Moment distribution

Denoting

\[
\exp(S_n x) = \exp \left( \begin{pmatrix}
T & -T & 0 & \cdots & 0 \\
0 & T & -T & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & T
\end{pmatrix}
\right) x,
\]

we get
The representation of a Moment distribution

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\[
\exp(S_n x) = \exp \left( \begin{pmatrix} T & -T & 0 & \cdots & 0 \\ 0 & T & -T & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & T \end{pmatrix} x \right),
\]

we get

\[
g_n(x) = \frac{\pi}{\mathbb{E}(\tau^n)} (-T)^{-n-1} (n)! (1, 0, \ldots, 0) \exp(S_{n+1} x) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -T \end{pmatrix} t \\
= \left( \frac{\pi T^{-n}}{\pi T^{-n} e}, 0, \ldots, 0 \right) \exp(S_{n+1} x) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ t \end{pmatrix}.
\]
The functional calculus method:
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- give us a straightforward way to obtain expressions like $\mathbb{E}(\omega(\tau))$. 
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- give us a straightforward way to obtain expressions like $\mathbb{E}(\omega(\tau))$.

- provides us a constructive way to obtain the representation of the moment distribution of a phase–type distributed random variable.