A Calabi operator for locally symmetric spaces (arxiv.org/abs/2112.00841)

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Setup

- Given a (semi-) Riemannian manifold (M, g) and its Levi-Civita connection ∇, the Killing operator is a map
$$\mathcal{K}\colon \text{ 1-tensors } \to \text{ symmetric 2-tensors } .$$
- One usually considers the 1-tensor as a vector field, and those annihilated by the Killing operator are exactly the infinitessimal generators of isometries of (M, g).
- However, it is congenial for us to ‘lower an index’ and speak of the (entirely equivalent) Killing operator on 1-forms,

$$\begin{tikzcd}
\bigwedge^1 \arrow[r] \arrow[rr, bend left, "\mathcal{K}"] & \bigwedge^1 \otimes \bigwedge^1 \arrow[r] & \bigodot^2 \bigwedge^1 \\
\sigma_a \arrow[r, mapsto] & \nabla_a \sigma_b \arrow[r] & \nabla_{(a}\sigma_{b)} = \frac{1}{2}\nabla_a \sigma_b + \frac{1}{2}\nabla_b \sigma_a
\end{tikzcd}$$

Abstract indices

- We employ Penrose's abstract index notation: σ_b denotes a 1-form because it has one lowered index. The connection applied to σ is $\nabla_a \sigma_b$, denoting a 1-form valued 1-form, aka a two tensor, a section of $\wedge^1 \otimes \wedge^1$.
- Given any tensor, say μ_{abc}, convenient to notate various symmetrizations, for example using $(\bullet, \bullet)$ for symmetrization in some indices and $[\bullet, \bullet]$ for skew symmetrization, e.g.:

$$\mu_{(ab)c} = \frac{1}{2}(\mu_{abc} + \mu_{bac}) \qquad \mu_{a[bc]} = \frac{1}{2}(\mu_{abc} - \mu_{acb})$$

- If you're not familiar with 'abstract indices', you can mentally replace e.g. μ_{abc} with 'the coefficients of the 3-tensor in a frame', so that $\mu = \mu_{abc} e^a e^b e^c$.

$$\begin{tikzcd}
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\end{tikzcd}$$

- The kernel of the Killing operator are infinitessimal isometries, so are generally well understood. A natural next question, to better understand the operator $\mathcal{K}$:

Question

What is the image of the Killing operator?

- You might expect that the Killing operator is so well studied that this is known in general, but it is not so.

Trivial metric perturbations

- One answer: the Killing operator has codomain $\{$symmetric 2-tensors$\}$, and the image 2-tensors $\mathcal{K}(X)$ correspond to trivial (gauge) perturbations of the background metric g,
$$g_{ab} \mapsto g_{ab} + \epsilon \mathcal{K}(X)_{ab}.$$
- In fact, a short calculation shows that for any vector field X, one has
$$\mathcal{L}_X g_{ab} = \mathcal{K}(X)_{ab}.$$
($\mathcal{L}$ the usual Lie derivative)
- To perturb the metric by $\mathcal{L}_X g_{ab}$—equivalently, an element in the image of $\mathcal{K}$—merely changes the metric up to infinitessimal diffeomorphism (by the integral flow of X). In other words, you've only changed coordinates, and not the metric itself.

Gravitational waves

IANAP[1], but this comes up for example in (linearized) gravitational waves, which are travelling non-trivial perturbations of the background metric. It is important to understand which perturbations are non-trivial, have physically observable effect.

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[1] I Am Not A Physicist

Calabi's operator

- To study the Killing equation on Riemannian manifolds of constant curvature, Calabi defined a second-order differential operator,
  $$\ydiagram{2} = \bigodot^2 \wedge^1 \ni h_{ab} \mapsto \mathcal{C}(h)_{abcd} \in \text{Riem} = \ydiagram{2,2},$$
  $$\begin{aligned}\mathcal{C}(h)_{abcd} =& \nabla_{(a}\nabla_{c)}h_{bd} - \nabla_{(b}\nabla_{c)}h_{ad} - \nabla_{(a}\nabla_{d)}h_{bc} + \nabla_{(b}\nabla_{d)}h_{ac} \\ & - R_{ab}{}^{e}{}_{[c}h_{d]e} - R_{cd}{}^{e}{}_{[a}h_{b]e}\end{aligned}$$
- Don't worry about the detailed structure here, but of import is that the 4-tensor $\mathcal{C}(h)_{abcd}$ has Riemannian curvature type symmetries (it is skew in *ab* and *cd*, plus the Bianchi-type symmetry: $\mathcal{C}(h)_{[abc]d} = 0$).

Heiroglyphs

- These ‘box diagrams’, *tableaux*, comprise an efficient and useful language for encoding the symmetries of tensors. There is an algorithm for reading off symmetries from the arrangement of boxes.
- But we only need four of these, so let’s just enumerate them:

$\ydiagram{1} =$ 1-tensors

$\ydiagram{2} =$ symmetric 2-tensors

$\ydiagram{1,1} =$ skew symmetric 2-tensors $=$ differential 2-forms

$\ydiagram{2,2} =$ 4-tensors of Riemannian symmetry

Composition of Killing and Calabi

- It is a few pages of calculation to check that the composition (on a general semi-Riemannian manifold)

  $$\begin{array}{|c|}\hline \\ \hline\end{array} \xrightarrow{\mathcal{K}} \begin{array}{|c|c|}\hline & \\ \hline\end{array} \xrightarrow{\mathcal{C}} \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array}$$

  has formula

  $$\sigma \mapsto R \cdot \, d\sigma - (\nabla R) \cdot \sigma,$$

  where the first term is the action of the two form $d\sigma$ on the background curvature, and the second term is contraction with σ.
- (In indices, more explicitly,

  $$2R_{ab}{}^{e}{}_{[c}\mu_{d]e} + 2R_{cd}{}^{e}{}_{[a}\mu_{b]e} - (\nabla^{e} R_{abcd})\sigma_{e}$$

  where $\mu_{ab} = \, d\sigma_{ab} = \nabla_{[a}\sigma_{b]}$.)

Constant curvature manifolds

Theorem, Calabi [1]

On a manifold of constant curvature, the complex is exact,

$$\ydiagram{1} \xrightarrow{\mathcal{K}} \ydiagram{2} \xrightarrow{\mathcal{C}} \ydiagram{2,2}$$

- Proof: On a manifold of constant curvature, the action of curvature on 2-forms is identically zero, and $\nabla R = 0$, so one finds that the composition has formula

$$\sigma \mapsto \cancel{R \cdot d\sigma} - \cancel{(\nabla R)} \cdot \sigma = 0.$$

Constant curvature manifolds

Theorem, Calabi [1]

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$$\ydiagram{1} \xrightarrow{\mathcal{K}} \ydiagram{2} \xrightarrow{\mathcal{C}} \ydiagram{2,2}$$

- Calabi gives an entire sequence of differential operators defining an exact sequence *resolving* the Killing operator,

$$0 \longrightarrow \ker \mathcal{K} \longrightarrow \ydiagram{1} \xrightarrow{\mathcal{K}} \ydiagram{2} \xrightarrow{\mathcal{C}} \ydiagram{2,2} \xrightarrow{\mathcal{C}'} \ydiagram{3,2} \longrightarrow \cdots$$

Locally symmetric manifolds

- Constant curvature is too restrictive an assumption! A bit more generally, a Riemannian manifold is *locally symmetric* if and only if $\nabla R = 0$ identically.
- The composition still simplifies,

$$\begin{tikzcd}
\ydiagram{1} \arrow[r, "\mathcal{K}"] & \ydiagram{2} \arrow[r, "\mathcal{C}"] & \ydiagram{2,2} \\
\sigma \arrow[rr, mapsto] & & R \cdot d\sigma + \cancel{\nabla R \cdot \sigma}
\end{tikzcd}$$

Not quite a complex, but...

- We have the operator ‘two forms acting on the curvature’, and the quotient

$$\ydiagram{1,1} \xrightarrow{R\cdot} \ydiagram{2,2} \xrightarrow{q} \overline{\ydiagram{2,2}} = \ydiagram{2,2} / R\left(\ydiagram{1,1}\right)$$

Locally symmetric manifolds

- So, if we modify the Calabi operator, we get a complex again! (On any locally symmetric manifold.)

$$\begin{tikzcd} \ydiagram{1} \arrow[r, "\mathcal{K}"] & \ydiagram{2} \arrow[r, "\overline{\mathcal{C}}"] & \overline{\ydiagram{2,2}} \\ \sigma \arrow[rr, mapsto] & & \cancel{R \cdot d}\sigma = 0 \end{tikzcd}$$

- Where we have defined

$$\overline{\mathcal{C}} : \ydiagram{2} \xrightarrow{\mathcal{C}} \ydiagram{2,2} \xrightarrow{q} \overline{\ydiagram{2,2}}$$

by quotienting out the image of 2-forms.

Is it locally exact?

Theorem, Costanza, Eastwood, Leistner, McMillan

Suppose M is a Riemannian locally symmetric space. If we write M as a product of irreducibles

$$M = M_1 \times M_2 \times \cdots \times M_k,$$

then the complex

$$\ydiagram{1} \xrightarrow{\mathcal{K}} \ydiagram{2} \xrightarrow{\bar{\mathcal{C}}} \overline{\ydiagram{2,2}}$$

is locally exact, except if M has at least one flat factor and at least one Hermitian factor, in which case it fails to be locally exact.

Whence the Calabi operator?

- Let's look more closely at the Killing operator for clues. Suppose that $\mathcal{K}(\sigma) = 0$, or equivalently, $\nabla_b\sigma_c = \nabla_{[b}\sigma_{c]} + \cancel{\mathcal{K}(\sigma)_{bc}} = d\sigma_{bc} =: \mu_{ab}$. This equation has a differential consequence:

$$\begin{aligned}\nabla_a\mu_{bc} &= \nabla_{[a}\mu_{b]c} - \nabla_{[a}\mu_{c]b} - \nabla_{[b}\mu_{c]a}\\ &= \nabla_{[a}\nabla_{b]}\sigma_c - \nabla_{[a}\nabla_{c]}\sigma_b - \nabla_{[b}\nabla_{c]}\sigma_a\\ &= \tfrac{1}{2}R_{ab}{}^{d}{}_{c}\sigma_d - \tfrac{1}{2}R_{ac}{}^{d}{}_{b}\sigma_d - \tfrac{1}{2}R_{bc}{}^{d}{}_{a}\sigma_d = R_{ab}{}^{d}{}_{c}\sigma_d\end{aligned}$$

- First equality is an identity on any 3 tensor skew in the last indices. (Like the one used to compute Christoffel symbols.) Second equality is the definition of μ_{ab}. The last equality is the Bianchi symmetry of curvature.

Prolongation of the Killing equation

- What we've found is that a 1-form σ_b is Killing if and only if there exists a 2-form μ_{bc} so that $\nabla_a \sigma_b - \mu_{ab} = 0$ *and* $\nabla_a \mu_{bc} - R_{ab}{}^d{}_c \sigma_d = 0$.
- Let $E = \wedge^1 \oplus \wedge^2$, and define a connection

$$E \ni \begin{bmatrix} \sigma_c \\ \mu_{cd} \end{bmatrix} \xmapsto{\mathcal{D}_b} \begin{bmatrix} \nabla_b \sigma_c - \mu_{bc} \\ \nabla_b \mu_{cd} - R_{cd}{}^e{}_b \sigma_e \end{bmatrix} \in \wedge^1 \otimes E$$

- Observe, σ_b is Killing if and only if $\mathcal{D}_b(\sigma_c, \, d\sigma_{cd}) = 0$. This is a prolongation of the Killing equation (which is an overdetermined equation), and it has closed up—a good situation to be in. We are able to replace the Killing operator with a connection on a larger bundle; solutions are in bijection with flat sections

Image of the prolonged equation

- For $E = \wedge^1 \oplus \wedge^2$ and the prolongation connection
$$E \ni \begin{bmatrix} \sigma_c \\ \mu_{cd} \end{bmatrix} \overset{\mathcal{D}_b}{\longmapsto} \begin{bmatrix} \nabla_b \sigma_c - \mu_{bc} \\ \nabla_b \mu_{cd} - R_{cd}{}^e{}_b \sigma_e \end{bmatrix} \in \wedge^1 \otimes E$$
it is not a difficult calculation to see that the curvature is given by
$$(\mathcal{D}_a \mathcal{D}_b - \mathcal{D}_b \mathcal{D}_a) \begin{bmatrix} \sigma_c \\ \mu_{cd} \end{bmatrix} = \begin{bmatrix} 0 \\ (R \cdot \mu)_{abcd} - (\nabla^e R_{abcd}) \sigma_e \end{bmatrix}$$
- This is exactly the composition $\overline{\mathcal{C}}\,\mathcal{K}$ that we saw before. (In particular, miraculously, the curvature of $\mathcal{D}$ has component only in the Riemannian symmetries component of $\wedge^2 \otimes \wedge^2$.) The $(\nabla^e R_{abcd})\sigma_e$ term vanishes on a locally symmetric manifold.

Image of the prolonged equation

- The prolongation connection is also related to the image of $\mathcal{K}$. If σ_c is *any* 1-form, then its image is $h_{bc} = \mathcal{K}(\sigma) = \nabla_{(b}\sigma_{c)} = \nabla_b\sigma_c - \nabla_{[b}\sigma_{c]}$, and so

$$\mathcal{D}_b \begin{bmatrix} \sigma_c \\ \nabla_{[c}\sigma_{d]} \end{bmatrix} = \begin{bmatrix} \nabla_b\sigma_c - \nabla_{[c}\sigma_{d]} \\ \nabla_b\nabla_{[c}\sigma_{d]} - R_{cd}{}^{e}{}_{b}\sigma_e \end{bmatrix} = \cdots = \begin{bmatrix} h_{bc} \\ 2\nabla_{[c}h_{d]b} \end{bmatrix}$$

- So, a symmetric h_{bc} is in the image of $\mathcal{K}$ if and only if the previous display holds for some σ_c.
- On the other hand, it's a direct computation that for any symmetric h_{bc},

$$\mathcal{D}_a \begin{bmatrix} h_{bc} \\ 2\nabla_{[c}h_{d]b} \end{bmatrix} - \mathcal{D}_b \begin{bmatrix} h_{ac} \\ 2\nabla_{[c}h_{d]a} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathcal{C}(h)_{abcd} \end{bmatrix}$$

Image of the prolonged equation

- Putting it all together, we have that *if* a symmetric h_{bc} is in the image of the Killing operator, then for some σ_a and $\mu_{ab} = \nabla_{[a}\sigma_{b]}$,

$$\begin{bmatrix} 0 \\ \mathcal{C}(h)_{abcd} \end{bmatrix} = (\mathcal{D}_a \mathcal{D}_b - \mathcal{D}_b \mathcal{D}_a) \begin{bmatrix} \sigma_c \\ \mu_{cd} \end{bmatrix} = \begin{bmatrix} 0 \\ (R \cdot \mu)_{abcd} \end{bmatrix}$$

- In other words, recalling that "$\overline{\mathcal{C}} = \mathcal{C}$ modulo $R \cdot \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array}$", we find that $\overline{\mathcal{C}}(h) = 0$ is a *necessary* condition for h_{bc} to be in the image of $\mathcal{K}$.
- Recall that I claimed it is a long calculation to compute the composition $\mathcal{C}\,\mathcal{K}$, but this re-does that in a few lines.

- The last couple slides got a little formula heavy, but the upshot is this: you can replace the Killing equation with a nice, geometrically adapted connection.
- It is then just a game of playing around with the connection to determine compatibility conditions to be in the image of the connection, equivalently the image of $\mathcal{K}$.
- This game can be played more generally, for other overdetermined linear operators!

Thanks for listening!

- E. Calabi, *On compact Riemannian manifolds with constant curvature I*, ‘Differential Geometry,’ Proc. Symp. Pure Math. vol. III, Amer. Math. Soc. 1961, pp. 155–180.
- F. Costanza, M. Eastwood, T. Leistner, B. McMIllan, *A Calabi operator for Riemannian locally symmetric spaces*, arXiv: 2112.00841