A Calabi operator for locally symmetric spaces (arxiv.org/abs/2112.00841)

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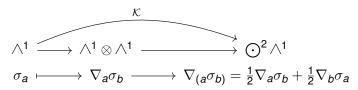
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Setup

Given a (semi-) Riemannian manifold (*M*, *g*) and its Levi-Civita connection ∇, the Killing operator is a map

 \mathcal{K} : 1-tensors \rightarrow symmetric 2-tensors .

- One usually considers the 1-tensor as a vector field, and those annihilated by the Killing operator are exactly the infinitessimal generators of isometries of (*M*, *g*).
- However, it is congenial for us to 'lower an index' and speak of the (entirely equivalent) Killing operator on 1-forms,



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Abstract indices

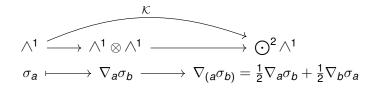
- We employ Penrose's abstract index notation: σ_b denotes a 1-form because it has one lowered index. The connection applied to σ is ∇_aσ_b, denoting a 1-form valued 1-form, aka a two tensor, a section of ∧¹ ⊗ ∧¹.
- Given any tensor, say µ_{abc}, convenient to notate various symmetrizations, for example using (•, •) for symmetrization in some indices and [•, •] for skew symmetrization, e.g.:

$$\mu_{(ab)c} = \frac{1}{2}(\mu_{abc} + \mu_{bac}) \qquad \mu_{a[bc]} = \frac{1}{2}(\mu_{abc} - \mu_{acb})$$

► If you're not familiar with 'abstract indices', you can mentally replace e.g. μ_{abc} with 'the coefficients of the 3-tensor in a frame', so that $\mu = \mu_{abc} e^a e^b e^c$.

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The kernel of the Killing operator are infinitessimal isometries, so are generally well understood. A natural next question, to better understand the operator K:

Question

What is the image of the Killing operator?

You might expect that the Killing operator is so well studied that this is known in general, but it is not so.

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Trivial metric perturbations

One answer: the Killing operator has codomain {symmetric 2-tensors}, and the image 2-tensors K(X) correspond to trivial (gauge) perturbations of the background metric g,

$$g_{ab} \mapsto g_{ab} + \epsilon \mathcal{K}(X)_{ab}.$$

In fact, a short calculation shows that for any vector field X, one has

$$\mathcal{L}_X g_{ab} = \mathcal{K}(X)_{ab}.$$

(\mathcal{L} the usual Lie derivative)

► To perturb the metric by L_Xg_{ab}—equivalently, an element in the image of K—merely changes the metric up to infinitessimal diffeomorphism (by the integral flow of X). In other words, you've only changed coordinates, and not the metric itself.

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Gravitational waves

IANAP¹, but this comes up for example in (linearized) gravitational waves, which are travelling non-trivial perturbations of the background metric. It is important to understand which perturbations are non-trivial, have physically observable effect.

¹I Am Not A Physicist

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Calabi's operator

 To study the Killing equation on Riemannian manifolds of constant curvature, Calabi defined a second-order differential operator,

$$\Box \Box = \bigcirc^2 \wedge^1 \ni h_{ab} \mapsto \mathcal{C}(h)_{abcd} \in \mathsf{Riem} = \Box,$$

$$\mathcal{C}(h)_{abcd} = \nabla_{(a} \nabla_{c)} h_{bd} - \nabla_{(b} \nabla_{c)} h_{ad} - \nabla_{(a} \nabla_{d)} h_{bc} + \nabla_{(b} \nabla_{d)} h_{ac} - R_{ab}^{e}{}_{[c} h_{d]e} - R_{cd}^{e}{}_{[a} h_{b]e}$$

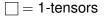
Don't worry about the detailed structure here, but of import is that the 4-tensor C(h)_{abcd} has Riemannian curvature type symmetries (it is skew in *ab* and *cd*, plus the Bianchi-type symmetry: C(h)_{[abc]d} = 0).

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Heiroglyphs

- These 'box diagrams', tableaux, comprise an efficient and useful language for encoding the symmetries of tensors. There is an algorithm for reading off symmetries from the arrangement of boxes.
- But we only need four of these, so let's just enumerate them:



- = skew symmetric 2-tensors = differential 2-forms

- = 4-tensors of Riemannian symmetry

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Composition of Killing and Calabi

 It is a few pages of calculation to check that the composition (on a general semi-Riemannian manifold)



has formula

$$\sigma \mapsto \boldsymbol{R} \cdot \boldsymbol{d}\sigma - (\nabla \boldsymbol{R}) \cdot \sigma,$$

where the first term is the action of the two form $d\sigma$ on the background curvature, and the second term is contraction with σ .

(In indices, more explicitly,

$$2R_{ab}{}^{e}{}_{[c}\mu_{d]e} + 2R_{cd}{}^{e}{}_{[a}\mu_{b]e} - (\nabla^{e}R_{abcd})\sigma_{e}$$

where $\mu_{ab} = d\sigma_{ab} = \nabla_{[a}\sigma_{b]}$.)

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Constant curvature manifolds

Theorem, Calabi [1]

On a manifold of constant curvature, the complex is exact,

$$\Box \xrightarrow{\mathcal{K}} \Box \xrightarrow{\mathcal{C}} \Box$$

Proof: On a manifold of constant curvature, the action of curvature on 2-forms is identically zero, and ∇R = 0, so one finds that the composition has formula

$$\sigma \mapsto \mathbf{B} \cdot \mathbf{\sigma} - (\nabla \mathbf{R}) \cdot \sigma = \mathbf{0}.$$

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Constant curvature manifolds

Theorem, Calabi [1]

On a manifold of constant curvature, the complex is exact,



 Calabi gives an entire sequence of differential operators defining an exact sequence *resolving* the Killing operator,

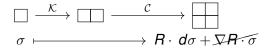
$$0 \longrightarrow \ker \mathcal{K} \longrightarrow \square \xrightarrow{\mathcal{K}} \square \xrightarrow{\mathcal{C}} \longrightarrow \square \xrightarrow{\mathcal{C}'} \longrightarrow \cdots$$

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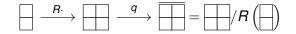
Locally symmetric manifolds

- Constant curvature is too restrictive an assumption! A bit more generally, a Riemannian manifold is *locally* symmetric if and only if ∇R = 0 identically.
- The composition still simplifies,



Not quite a complex, but...

We have the operator 'two forms acting on the curvature', and the quotient

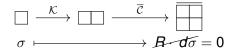


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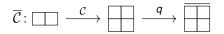
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Locally symmetric manifolds

So, if we modify the Calabi operator, we get a complex again! (On any locally symmetric manifold.)



Where we have defined



by quotienting out the image of 2-forms.

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Is it locally exact?

Theorem, Costanza, Eastwood, Leistner, McMillan Suppose M is a Riemannian locally symmetric space. If we write M as a product of irreducibles

$$M = M_1 \times M_2 \times \cdots \times M_k,$$

then the complex



is locally exact, except if M has at least one flat factor and at least one Hermitian factor, in which case it fails to be locally exact.

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Whence the Calabi operator?

• Let's look more closely at the Killing operator for clues. Suppose that $\mathcal{K}(\sigma) = 0$, or equivalently, $\nabla_b \sigma_c = \nabla_{[b} \sigma_{c]} + \mathcal{K}(\sigma)_{bc} = d\sigma_{bc} =: \mu_{ab}$. This equation has a differential consequence:

$$\begin{aligned} \nabla_{a}\mu_{bc} &= \nabla_{[a}\mu_{b]c} - \nabla_{[a}\mu_{c]b} - \nabla_{[b}\mu_{c]a} \\ &= \nabla_{[a}\nabla_{b]}\sigma_{c} - \nabla_{[a}\nabla_{c]}\sigma_{b} - \nabla_{[b}\nabla_{c]}\sigma_{a} \\ &= \frac{1}{2}R_{ab}{}^{d}{}_{c}\sigma_{d} - \frac{1}{2}R_{ac}{}^{d}{}_{b}\sigma_{d} - \frac{1}{2}R_{bc}{}^{d}{}_{a}\sigma_{d} = R_{ab}{}^{d}{}_{c}\sigma_{d} \end{aligned}$$

First equality is an identity on any 3 tensor skew in the last indices. (Like the one used to compute Christoffel symbols.) Second equality is the definition of μ_{ab}. The last equality is the Bianchi symmetry of curvature.

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Prolongation of the Killing equation

- What we've found is that a 1-form σ_b is Killing if and only if there exists a 2-form μ_{bc} so that $\nabla_a \sigma_b \mu_{ab} = 0$ and $\nabla_a \mu_{bc} R_{ab}{}^d{}_c \sigma_d = 0$.
- Let $E = \wedge^1 \oplus \wedge^2$, and define a connection

$$\boldsymbol{E} \ni \begin{bmatrix} \sigma_{\boldsymbol{c}} \\ \mu_{\boldsymbol{cd}} \end{bmatrix} \stackrel{\mathcal{D}_{\boldsymbol{b}}}{\longmapsto} \begin{bmatrix} \nabla_{\boldsymbol{b}} \sigma_{\boldsymbol{c}} - \mu_{\boldsymbol{bc}} \\ \nabla_{\boldsymbol{b}} \mu_{\boldsymbol{cd}} - \boldsymbol{R_{\boldsymbol{cd}}}^{\boldsymbol{e}}{}_{\boldsymbol{b}} \sigma_{\boldsymbol{e}} \end{bmatrix} \in \wedge^{1} \otimes \boldsymbol{E}$$

Observe, σ_b is Killing if and only if D_b(σ_c, dσ_{cd}) = 0. This is a prolongation of the Killing equation (which is an overdetermined equation), and it has closed up—a good situation to be in. We are able to replace the Killing operator with a connection on a larger bundle; solutions are in bijection with flat sections

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Image of the prolonged equation

• For $E = \wedge^1 \oplus \wedge^2$ and the prolongation connection

$$\boldsymbol{E} \ni \begin{bmatrix} \sigma_{\boldsymbol{c}} \\ \mu_{\boldsymbol{cd}} \end{bmatrix} \stackrel{\mathcal{D}_{\boldsymbol{b}}}{\longmapsto} \begin{bmatrix} \nabla_{\boldsymbol{b}} \sigma_{\boldsymbol{c}} - \mu_{\boldsymbol{b}\boldsymbol{c}} \\ \nabla_{\boldsymbol{b}} \mu_{\boldsymbol{cd}} - \boldsymbol{R_{\boldsymbol{cd}}}^{\boldsymbol{e}}{}_{\boldsymbol{b}} \sigma_{\boldsymbol{e}} \end{bmatrix} \in \wedge^{1} \otimes \boldsymbol{E}$$

it is not a difficult calculation to see that the curvature is given by

$$\left(\mathcal{D}_{a}\mathcal{D}_{b}-\mathcal{D}_{b}\mathcal{D}_{a}\right)\left[\begin{array}{c}\sigma_{c}\\\mu_{cd}\end{array}\right]=\left[\begin{array}{c}0\\(R\cdot\mu)_{abcd}-(\nabla^{e}R_{abcd})\sigma_{e}\end{array}\right]$$

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Image of the prolonged equation

- The prolongation connection is also related to the image of \mathcal{K} . If σ_c is any 1-form, then its image is $h_{bc} = \mathcal{K}(\sigma) = \nabla_{(b}\sigma_{c)} = \nabla_{b}\sigma_{c} \nabla_{[b}\sigma_{c]}$, and so $\mathcal{D}_b \begin{bmatrix} \sigma_c \\ \nabla_{[c}\sigma_{d]} \end{bmatrix} = \begin{bmatrix} \nabla_b \sigma_c \nabla_{[c}\sigma_{d]} \\ \nabla_b \nabla_{[c}\sigma_{d]} R_{cd}{}^e{}_b \sigma_e \end{bmatrix} = \dots = \begin{bmatrix} h_{bc} \\ 2\nabla_{[c}h_{d]b} \end{bmatrix}$
- So, a symmetric h_{bc} is in the image of *K* if and only if the previous display holds for some σ_c.
- On the other hand, it's a direct computation that for any symmetric h_{bc},

$$\mathcal{D}_{a}\left[\begin{array}{c}h_{bc}\\2\nabla_{[c}h_{d]b}\end{array}\right] - \mathcal{D}_{b}\left[\begin{array}{c}h_{ac}\\2\nabla_{[c}h_{d]a}\end{array}\right] = \left[\begin{array}{c}0\\\mathcal{C}(h)_{abcd}\end{array}\right]$$

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Image of the prolonged equation

Putting it all together, we have that *if* a symmetric h_{bc} is in the image of the Killing operator, then for some σ_a and μ_{ab} = ∇_{[a}σ_{b]},

$$\begin{bmatrix} 0\\ \mathcal{C}(h)_{abcd} \end{bmatrix} = (\mathcal{D}_{a}\mathcal{D}_{b} - \mathcal{D}_{b}\mathcal{D}_{a})\begin{bmatrix} \sigma_{c}\\ \mu_{cd} \end{bmatrix} = \begin{bmatrix} 0\\ (R \cdot \mu)_{abcd} \end{bmatrix}$$

- In other words, recalling that "C̄ = C modulo R · _, we find that C̄(h) = 0 is a *necessary* condition for h_{bc} to be in the image of K.
- Recall that I claimed it is a long calculation to compute the composition C K, but this re-does that in a few lines.

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- The last couple slides got a little formula heavy, but the upshot is this: you can replace the Killing equation with a nice, geometrically adapted connection.
- It is then just a game of playing around with the connection to determine compatibility conditions to be in the image of the connection, equivalently the image of K.
- This game can be played more generally, for other overdetermined linear operators!

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Thanks for listening!

- E. Calabi, On compact Riemannian manifolds with constant curvature I, 'Differential Geometry,' Proc. Symp. Pure Math. vol. III, Amer. Math. Soc. 1961, pp. 155–180.
- F. Costanza, M. Eastwood, T. Leistner, B. McMIllan, *A Calabi operator for Riemannian locally symmetric spaces*, arXiv: 2112.00841

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