

RIEMANNIAN HOLONOMY

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Note: Much of this is a distillation of the treatment of holonomy in the book *Einstein Manifolds* of Arthur L. Besse, for more detail the reader is strongly encouraged to look at the original source. This talk covers the case of holonomy on Riemannian manifolds, but there is an analogous theory for manifolds with more general connections.

1. WHAT IS HOLONOMY?

Given a connected Riemannian manifold M with Levi-Civita connection ∇ we may define the parallel transport along a curve. That is to say, for any curve $\gamma: [0, 1] \rightarrow M$ and a vector $X_0 \in T_{\gamma(0)}M$ there is a unique parallel vector field X extending X_0 along γ so that $\nabla_{\gamma'}X \equiv 0$. The *parallel transport* of X_0 is then the vector $X(1) \in T_{\gamma(1)}M$. By varying X_0 we get a map $P_\gamma: T_{\gamma(0)}M \rightarrow T_{\gamma(1)}M$. If γ is a closed loop based at xp then we get an endomorphism of T_pM . Varying these loops (but fixing p), we have a functor from the loop space $\Omega(M, p)$ to $\text{End}(T_pM)$, which is a quick way of saying that for 2 loops γ and η we have $P_{\gamma \cdot \eta} = P_\gamma P_\eta$. Now we can define

Definition. The holonomy group of M at p , $\text{Hol}(M, p)$ is the image of $\Omega(M, p)$ under the functor described above with group law given by composition.

To see that $\text{Hol}(M, p)$ is actually a group note that the functoriality $P_\gamma P_\eta = P_{\gamma \cdot \eta}$ shows both that $\text{Hol}(M, p)$ is closed under composition and the image of the constant loop is the identity. To see that every element has an inverse note that the uniqueness of parallel vector field extending any vector implies that $P_{\gamma^{-1}} = P_\gamma^{-1}$.

In fact we can say more. Recall that our parallel transport map comes from the solution of a homogenous ODE on the tangent bundle. In particular, the solution depends linearly on the initial condition X_0 , so $\text{Hol}(M, p)$ is contained in $GL(T_pM)$. Furthermore, since the metric on M is compatible with ∇ , the map P_γ must be an isometry of T_pM , so that $\text{Hol}(p, M) \subset O(T_pM)$. Finally, we note that the holonomy lies in $SO(T_pM)$ exactly if M is orientable.

Now, the definition as stated depends on the chosen base point p , but consider another point $q \in M$ and any path η from p to q . Then the map from $\text{Hol}(M, p)$ to $\text{Hol}(M, q)$ given by conjugating with P_η is an isomorphism. Thus we can drop the base point from our notation and talk of the holonomy of M .

Example. If $M = \mathbb{R}^n$ with the flat metric then parallel translation is simply translation in \mathbb{R}^n . In particular, P_γ is the identity for each γ , so $\text{Hol}(\mathbb{R}^n) = 0$.

If M is flat it is not necessarily true that $\text{Hol}(M) = 0$. For example, consider the cone of angle α around the vertex. By unrolling the cone it is not hard to see that the holonomy of any simple loop around the vertex will be rotation by the angle α . The holonomy group is generated by this rotation and will be finite or infinite cyclic depending on whether α/π is rational or irrational.

Note however that the holonomy is discrete. This is the worst that can happen for a flat manifold. In fact for any flat manifold M the Lie algebra of $\text{Hol}(M)$ is zero. This will follow immediately from the Ambrose-Singer theorem, which states roughly that the Lie algebra of the holonomy is generated by the curvature at each point.

Example. The round sphere S^2 has holonomy group all of $SO(2)$. Each rotation $R_\alpha \in SO(2)$ is the holonomy of the following triangle based at the north pole: follow any geodesic to the equator, travel along the equator, return to the north pole along the geodesic at angle α with the one leaving the north pole. Since a parallel vector field along a geodesic keeps constant angle with the velocity of the geodesic (metric compatibility of the connection) we see that the holonomy of this loop is rotation by α .

2. HOLONOMY AND GEOMETRY OF A MANIFOLD

Suppose we have a tensor field on our manifold which has zero covariant derivative. If the tensor field is geometrically significant then this means we have a structure which is compatible with our metric. For example, a Kahler manifold is characterized by the fact that its complex structure has zero covariant derivative. The following theorem demonstrates the relation between holonomy and the geometry of a Riemannian manifold.

Theorem. *On a Riemannian manifold M the following are equivalent:*

- (1) *there exists a tensor field of type (r, s) which is parallel (invariant under parallel transport).*¹
- (2) *there exists a tensor field of type (r, s) with zero covariant derivative.*
- (3) *for some p in M there is an (r, s) tensor on T_pM which is invariant under $\text{Hol}(M, x)$.*

Proof. Suppose we have a tensor T which satisfies (1), so that for any path γ from p to q we have $P_\gamma T_p = T_q$. Then for any closed loop at p we have $P_\gamma T_p = T_p$, which implies (3). Conversely, for a tensor T_0 at p satisfying (3) we define $T_q = P_\gamma T_p$. Since T_0 is invariant under holonomy we see that T_q is independent of the choice of γ .

To see that (1) and (2) are equivalent recall the formula

$$(\nabla T)(X_1, \dots, X_s, X) = \nabla_X(T(X_1, \dots, X_s)) - \sum_{i=1}^s T(X_1, \dots, \nabla_X X_i, \dots, X_s).$$

We vary γ and choose X_1, \dots, X_s parallel along γ and let $X = \dot{\gamma}$. Since $\nabla_X X_i = 0$ we see that $\nabla T = 0$ if and only if $T(X_1, \dots, X_s)$ is constant along all paths. \square

Example. A Kahler manifold is a complex manifold (M^{2n}, g, J) which satisfies $\nabla J = 0$. By the theorem this means that $\text{Hol}(M)$ preserves the complex structure, or in other words $\text{Hol}(M) \subset GL(T_pM) \cong GL(\mathbb{C}^m)$ with the complex structure on T_pM given by J . Thus $\text{Hol}(M) \subset U(m) = SO(2m) \cap GL(\mathbb{C}^m)$.

Conversely, if $\text{Hol}(M) \subset U(n)$ then we fix J_0 a complex structure on T_pM preserved by $\text{Hol}(M)$. By the theorem this gives us an almost complex structure J on M which is furthermore parallel. It can be shown that this gives an honest complex structure on M , with which M is a Kahler manifold.

3. HOLONOMY AND CURVATURE

To see how curvature generates holonomy, fix vectors $X_0, Y_0 \in T_pM$. We extend these to commuting vector fields X, Y near p . Then the curvature $R(X, Y)$ has the nice form $[\nabla_X, \nabla_Y]$. Recall that the geometric interpretation of $[X, Y]$ for arbitrary vector fields is the derivative of the endpoint of the ‘parallelograms’ γ_t created by following the flow of X for time \sqrt{t} , of Y for \sqrt{t} , the inverse flow of X and then the inverse flow of Y . In our case, $[X, Y] = 0$ implies that γ_t is a closed loop for each t (at least up to first order, and

¹A parallel transport map is a linear isomorphism, so may be extended to tensors. For example, given a 1 form η on M and γ from x to y we have $(P_\gamma \eta_x)(v) = \eta_x(P_\gamma^{-1}v)$. In particular, a tensor field is parallel exactly if it is constant on all parallel vector fields.

we could modify our extensions to make it literally true.). The same proof with $[\nabla_X, \nabla_Y]$ shows that

$$R(X_0, Y_0) = [\nabla_X, \nabla_Y] = \left. \frac{d}{dt} \right|_{t=0} P_{\gamma_t}.$$

In particular, we see that the sub algebra generated by $R(X, Y)$ as we vary $X, Y \in T_p M$ as well as $T_p M$ lies in the Lie algebra of $\text{Hol}(M)$. What is possibly more surprising is that this is all of the holonomy

Theorem (Ambrose-Singer). *The Lie algebra of $\text{Hol}(M, p)$ is generated by all elements of the form $P_\gamma^{-1} R(x, y) P_\gamma$ where γ is a path from p to q and $x, y \in T_q M$.*

An immediate consequence is

Corollary. *A generic oriented Riemannian manifold has holonomy group all of $SO(n)$.*

4. SYMMETRIC SPACES

The majority of spaces with nontrivial holonomy are locally symmetric spaces. To define these, note that at a point p in a Riemannian manifold M we may define a diffeomorphism on a neighborhood of p by flipping all of the geodesics through p . A space is locally symmetric if this geodesic inversion around any point is an isometry.

Example. Consider a Lie group G with an involution σ of G . If a compact subgroup H is an open subset of the fixed point set of σ then the manifold $M = G/H$ is locally symmetric.

The Lie group homomorphism σ induces a linear map $\sigma'_e: \mathfrak{g} \rightarrow \mathfrak{g}$ which is also an involution. Thus \mathfrak{g} decomposes into the direct sum of its $+1$ eigenspace \mathfrak{h} and the -1 eigenspace \mathfrak{m} . Since H is an open subset of $\text{Fix}(\sigma)$ we see that \mathfrak{h} is the Lie algebra of H and so \mathfrak{m} may be identified with the tangent space of M at eH . We fix a Euclidean structure on \mathfrak{m} (considered as the tangent space of M) which is invariant under the adjoint action of H and push it around by the action of G . The involution σ is an isometry, and the fact that σ'_e is the negative of the identity shows that σ is a geodesic inversion. Similarly, $g\sigma g^{-1}$ will be an isometric geodesic inversion at any other point gH .

Symmetric spaces have particularly easy to compute holonomy, as given by the following theorem

Theorem. *An irreducible simply connected symmetric space G/H has holonomy group isomorphic to H .*

The irreducible simply connected symmetric spaces were classified by Cartan, although it is a difficult task.

5. CLASSIFICATION OF NONTRIVIAL, NON-SYMMETRIC HOLONOMY

To complete the classification of possible holonomy groups we need to classify the possibilities if M is not locally symmetric. This involves a lot of representation theory, but the conclusion is that

Theorem. *Suppose M is a simply connected Riemannian manifold of dimension n whose representation of $\text{Hol}(M)$ on any tangent plane is irreducible. If M is not locally symmetric then the Lie group of $\text{Hol}(M)$ is one of*

- (1) $SO(n)$
- (2) $U(m)$, with $n = 2m$
- (3) $SU(m)$, with $n = 2m$

- (4) $Sp(1) \cdot Sp(m)$, with $n = 4m$
- (5) $Sp(m)$, with $n = 4m$
- (6) $Spin(7)$
- (7) $Spin(9)$
- (8) G_2

As remarked before, case 2 corresponds to a Kahler manifold. The other cases correspond to other kinds of geometries compatible with the metric.