# EVERY PROJECTIVE OKA MANIFOLD IS ELLIPTIC

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ABSTRACT. We show that every projective Oka manifold is elliptic in the sense of Gromov. This gives an affirmative answer to a long-standing open question.

## 1. INTRODUCTION

A complex manifold $Y$ is said to be an Oka manifold if it satisfies all forms of the homotopy principle (also called Oka principle in this context) for holomorphic maps $X \to Y$ from any Stein manifold $X$. One of the simplest characterisations of this class of manifolds is the convex approximation property introduced in [9]; see also [11, Sect. 5.4]. In Gromov's terminology [16, 3.1, p. 878], Oka manifolds are called $\mathrm{Ell}_\infty$ manifolds.

A complex manifold $Y$ is said to be elliptic if it admits a dominating holomorphic spray $s : E \to Y$ defined on the total space of a complex vector bundle $\pi : E \to Y$ [16, 0.5, p. 855]. This means that $s$ restricts to the identity map on the zero section $E_0 \cong Y$ of $E$, and for every $y \in Y$, the differential $ds_{0_y}$ at the origin $0_y \in E_y = \pi^{-1}(y)$ maps the fibre $E_y$ onto $T_yY$. An ostensibly weaker condition, called subellipticity, was introduced by the first-named author in [7, Definition 2]. It asks for the existence of finitely many holomorphic sprays $(E_j, \pi_j, s_j)$ on $Y$ $(j = 1, \ldots, m)$ that are dominating in the sense that

$$(ds_1)_{0_y}(E_{1,y}) + (ds_2)_{0_y}(E_{2,y}) + \cdots + (ds_m)_{0_y}(E_{m,y}) = T_yY \quad \text{for all } y \in Y. \tag{1.1}$$

One of the main results of Oka theory is that every elliptic manifold is an Oka manifold (see Gromov [16, 0.6, p. 855] and [14]), and every subelliptic manifold is an Oka manifold (see [7, Theorem 1.1]). For a comprehensive survey, see [11, Chap. 5]. Examples of elliptic and subelliptic manifolds can be found in [11, Sect. 6.4] and in the surveys [6, 10, 13]. Every complex homogeneous manifold is elliptic but the converse fails in general. Several recent results are mentioned below.

In this paper we prove the following main result.

**Theorem 1.1.** *Every projective Oka manifold is elliptic.*

It follows that, for projective manifolds, the Oka property, ellipticity, and subellipticity are equivalent. The spray bundle on a projective Oka manifold $Y$ that emerges in the proof of the theorem is easily described: it is the direct sum of some number of copies of the dual of a sufficiently ample line bundle on $Y$. In a suitable embedding of $Y$ into complex projective space, the spray bundle is therefore the direct sum of copies of the universal line bundle.

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*Date*: 29 March 2025.

2020 *Mathematics Subject Classification.* Primary 32Q56; secondary 32E10.

*Key words and phrases.* Oka manifold, elliptic manifold, subelliptic manifold, projective manifold, Stein manifold.

Theorem 1.1 solves a long-standing open problem, originating in Gromov's 1989 paper [16, 3.2.A" Question], whether every Oka manifold is elliptic; see also [11, Problem 6.4.21], where the analogous question was posed for subellipticity. The first counterexamples to both questions for noncompact manifolds were found only very recently. In 2024, Kusakabe showed that the complement $\mathbb{C}^n \setminus K$ of any compact polynomially convex set $K \subset \mathbb{C}^n$ for $n \geq 2$ is an Oka manifold [22, Theorem 1.6]. A few years earlier it was shown by Andrist, Shcherbina, and Wold [2] that if $K$ is a compact set with nonempty interior in $\mathbb{C}^n$ for $n \geq 3$, then $\mathbb{C}^n \setminus K$ fails to be subelliptic. Taking $K$ to also be polynomially convex, it follows that $\mathbb{C}^n \setminus K$ is Oka but not subelliptic. These examples are not Stein. As observed by Gromov, every Stein Oka manifold is elliptic [16, 3.2.A, p. 879].

In light of Theorem 1.1, the main remaining question on this topic is the following.

**Problem 1.2.** Is there a compact non-projective Oka manifold that fails to be elliptic or subelliptic?

A much-studied property of algebraic manifolds is the algebraic version of ellipticity. A complex algebraic manifold $Y$ is said to be algebraically elliptic if it admits an algebraic dominating spray $s : E \to Y$ defined on the total space of an algebraic vector bundle $\pi : E \to Y$; see [11, Definition 5.6.13 (e)]. Similarly, $Y$ is algebraically subelliptic if it admits finitely many algebraic sprays $(E_j, \pi_j, s_j)$ satisfying (1.1). It was recently shown by Kaliman and Zaidenberg [20] that every algebraically subelliptic manifold is algebraically elliptic; the converse is a tautology. Algebraic ellipticity is a Zariski-local condition as shown by Gromov [16, 3.5.B, 3.5.C]; see also [11, Proposition 6.4.2]. No such results are known in the holomorphic category. The optimal known geometric sufficient condition for a compact algebraic manifold to be algebraically elliptic is uniform rationality; see Arzhantsev, Kaliman, and Zaidenberg [3, Theorem 1.3]. See also the recent papers [18, 19] and the survey [29]. Every algebraically elliptic manifold $Y$ satisfies the algebraic homotopy approximation theorem for maps $X \to Y$ from affine algebraic manifolds $X$, showing in particular that every holomorphic map that is homotopic to an algebraic map is a limit of algebraic maps in the compact-open topology; see [8, Theorem 3.1], [11, Theorem 6.15.1], and the recent generalisations in [1, Sect. 2]. As shown by Lárusson and Truong [24], this is the closest analogue of the Oka principle in the algebraic category. However, there are examples of projective Oka manifolds that fail to be algebraically elliptic, for example, abelian varieties. Hence, the algebraic counterpart to Theorem 1.1 is not true, and the GAGA principle of Serre [28] fails for ellipticity of projective manifolds.

Besides its intrinsic importance in Oka theory, Theorem 1.1 is interesting for the following reason. For a long time, essentially the only known examples of Oka manifolds were the elliptic and subelliptic manifolds. Thanks to recent developments, we now have at our disposal several other methods to discover Oka manifolds. In particular, Kusakabe's localisation theorem [21, Theorem 1.4] says that a complex manifold covered by Zariski-open (in the holomorphic sense) Oka domains is an Oka manifold. No such localisation result is available for holomorphic ellipticity or subellipticity, so it is interesting that we nevertheless get ellipticity from the Oka property in the class of projective manifolds.

## 2. Proof of Theorem 1.1

Let $Y$ be a projective manifold, embedded in $n$-dimensional complex projective space $\mathbb{CP}^n$. Let $z = [z_0 : z_1 : \cdots : z_n]$ be homogeneous coordinates on $\mathbb{CP}^n$. Set $\Lambda_\alpha = \{z_\alpha = 0\}$ for $\alpha = 0, 1, \ldots, n$, and let $U_\alpha = \mathbb{CP}^n \setminus \Lambda_\alpha \cong \mathbb{C}^n$ be the affine chart with coordinates $(z_0/z_\alpha, \ldots, z_n/z_\alpha)$, where the term $z_\alpha/z_\alpha = 1$ is omitted. Denote the affine coordinates on $U_0$ by $x = (x_1, \ldots, x_n)$, with $x_i = z_i/z_0$. Let $\partial_{x_i} = \partial/\partial x_i$ denote the coordinate vector fields on $U_0 \cong \mathbb{C}^n$. Since $Y_0 = Y \cap U_0$ is an algebraic submanifold of $U_0 \cong \mathbb{C}^n$, Serre's Theorem A gives finitely many polynomial vector fields

$$W_j(x) = \sum_{i=1}^{n} V_{i,j}(x)\partial_{x_i}, \quad j = 1, \ldots, m, \tag{2.1}$$

on $U_0 \cong \mathbb{C}^n$ whose restrictions to $Y_0$ are tangent to $Y_0$ and span the tangent space $T_yY$ at every point $y \in Y_0$. To this collection we associate the polynomial vector field $V$ on the total space $U_0 \times \mathbb{C}^m \cong \mathbb{C}^{n+m}$ of the trivial vector bundle $\pi : U_0 \times \mathbb{C}^m \to U_0$, defined by

$$V(x,t) = \sum_{i=1}^{n}\sum_{j=1}^{m} t_j\, V_{i,j}(x)\partial_{x_i}, \tag{2.2}$$

where $x \in U_0$ and $t = (t_1, \ldots, t_m) \in \mathbb{C}^m$ is the fibre variable. Note that $V$ is horizontal in the sense that its $t$-component equals zero. Furthermore, $V$ vanishes on the zero section $U_0 \times \{0\}^m = \{t = 0\}$, and for every $(x,t) \in Y_0 \times \mathbb{C}^m$,

$$d\pi_{(x,t)}V(x,t) = \sum_{i=1}^{n}\sum_{j=1}^{m} t_j\, V_{i,j}(x)\partial_{x_i} = \sum_{j=1}^{m} t_j\, W_j(x) \in T_xY. \tag{2.3}$$

The formula without the last inclusion holds for all $(x,t) \in U_0 \times \mathbb{C}^m$.

Denote by $\mathbb{U} \to \mathbb{CP}^n$ the universal line bundle. Recall that $\Lambda_0 = \mathbb{CP}^n \setminus U_0 = \{z_0 = 0\}$.

**Lemma 2.1.** *There is an integer $k_0$ that only depends on the degrees of the polynomials $V_{i,j}(x)$, such that for every $k \geq k_0$, the vector field $V$* (2.2) *extends to an algebraic vector field on the total space of the vector bundle $E = (\mathbb{CP}^n \times \mathbb{C}^m) \otimes \mathbb{U}^k = m\mathbb{U}^k$ on $\mathbb{CP}^n$ (the direct sum of $m$ copies of the $k$-th tensor power of $\mathbb{U}$), which vanishes on the zero section $E_0$ of $E$ and on $E|_{\Lambda_0}$.*

*Proof.* For every $\alpha = 0, 1, \ldots, n$, we have a line bundle trivialisation $\theta_\alpha : E|U_\alpha \xrightarrow{\cong} U_\alpha \times \mathbb{C}^m$ with transition maps $\theta_{\alpha,\beta} = \theta_\alpha \circ \theta_\beta^{-1}$ on $(U_\alpha \cap U_\beta) \times \mathbb{C}^m$ given by

$$\theta_{\alpha,\beta}([z], t) = \big([z], (z_\alpha/z_\beta)^k t\big), \quad t \in \mathbb{C}^m,\ 0 \leq \alpha, \beta \leq n.$$

In particular, $\theta_{\alpha,0}([z], t) = \big([z], (z_\alpha/z_0)^k t\big)$. We analyse the behaviour of the vector field $V$ (2.2) near the hyperplane $\Lambda_0 \setminus \Lambda_\alpha$ for $\alpha = 1, \ldots, n$. It suffices to consider the case $\alpha = 1$ since the same argument will apply to every $\alpha$. Replacing the first coordinate $x_1 = z_1/z_0$ by $x_1' = 1/x_1 = z_0/z_1$, the vector field $V$ has the same form (2.2), where the coefficient functions $V_{i,j}'(x_1', x_2, \ldots, x_n)$ are rational with poles along the hyperplane $\{x_1' = 0\} = \{z_0 = 0\}$. Note that $\partial_{x_1'} = -(x_1')^{-2}\partial_{x_1} = -x_1^2\partial_{x_1}$, so we have $V_{1,j}'(x_1', x_2, \ldots, x_n) = -x_1^{-2}V_{1,j}(x_1, \ldots, x_n)$ and $V_{i,j}'(x_1', x_2, \ldots, x_n) = V_{i,j}(x_1, \ldots, x_n)$ for $i \neq 1$. To simplify the notation, we now drop the tildes, so $x_1 = z_0/z_1$. In these coordinates, the transition map $\theta_{1,0}$ is given by $\theta_{1,0}(x,t) = \big(x, x_1^{-k}t\big)$. Its differential has the block form

$$D\theta_{1,0}(x,t) = \begin{pmatrix} I_n & 0 \\ B & x_1^{-k} I_m \end{pmatrix}$$

where $I_n$ is the $n \times n$ identity matrix and $B = (-kx_1^{-k-1}t, 0, \ldots, 0)$ is an $m \times n$ matrix. Hence, the image vector field $V' = (\theta_{1,0})_* V$ on the chart $E|U_1$ for $x \in U_0 \cap U_1$ equals

$$V'(x,t) = D\theta_{1,0}(x,t)V(x,t) = \sum_{i=1}^{n}\sum_{j=1}^{m} t_j\, V_{i,j}(x)\partial_{x_i} - kx_1^{-k-1}\sum_{j,l=1}^{m} t_j t_l V_{1,j}(x)\partial_{t'_l}.$$

In terms of the new fibre variable $t' = x_1^{-k}t$ (so $t = x_1^k t'$), we have

$$V'(x,t') = \sum_{i=1}^{n}\sum_{j=1}^{m} t'_j\, x_1^k\, V_{i,j}(x)\partial_{x_i} - kx_1^{k-1}\sum_{j,l=1}^{m} t'_j t'_l\, V_{1,j}(x)\partial_{t'_l}. \tag{2.4}$$

For $k \geq k_0 = \max_{i,j} \deg V_{i,j} + 2$, the vector field $V'$ extends to the points of $E$ over the hyperplane $\Lambda_0 \setminus \Lambda_1 = \{x_1 = 0\}$ and vanishes there. Applying this argument for every $\alpha = 1, \ldots, n$, we see that for $k$ as above the vector field $V$ (2.2) extends to the vector bundle $E = m\mathbb{U}^k$ and vanishes on $E_0 \cup (E|\Lambda_0)$. $\square$

Since the extended vector field $V$ on $E$, given by Lemma 2.1, vanishes on the zero section $E_0$ of $E$, there is a neighbourhood $\Omega \subset E$ of $E_0$ such that the flow $\phi_\tau(e)$ of $V$, starting for $\tau = 0$ at any point $e \in \Omega$, exists for all $\tau \in [0,1]$. We may assume that $\Omega$ has convex fibres. The map

$$s = \pi \circ \phi_1 : \Omega \to \mathbb{CP}^n$$

is then a local holomorphic spray on $\mathbb{CP}^n$. On the zero section $E_0 \cong \mathbb{CP}^n$ we have a natural splitting $TE|E_0 = E \oplus T\mathbb{CP}^n$. Identifying a vector $e \in E_x = \pi^{-1}(x)$ with $e \in T_{0_x}E_x$, we let

$$(Vds)_x(e) = (ds)_{0_x}(e) \in T_x\mathbb{CP}^n$$

denote the vertical derivative of $s$ at $x$ applied to the vector $e$. We claim that for every $e = (x,t) \in \Omega$ with $x \in U_0$, we have

$$(Vds)_x(t_1, \ldots, t_m) = \sum_{j=1}^{m} t_j\, W_j(x). \tag{2.5}$$

To see this, note that in the vector bundle chart on $E|U_0$, the vector field $V$ is of the form (2.2), that is, it is horizontal and its coefficients are linear in the fibre variable $t$. It follows that

$$\pi \circ \phi_\tau(x, \delta t) = \pi \circ \phi_{\delta\tau}(x,t) \tag{2.6}$$

for every $(x,t) \in E|U_0$, $0 \leq \delta \leq 1$, and all $\tau$ for which the flow exists. Taking $(x,t) \in \Omega$, this holds for all $\tau \in [0,1]$. At $\tau = 1$ we obtain

$$s(x,\delta t) = \pi \circ \phi_1(x,\delta t) = \pi \circ \phi_\delta(x,t), \quad 0 \leq \delta \leq 1.$$

Differentiating with respect to $\delta$ at $\delta = 0$ and noting that $\frac{d}{d\delta}\Big|_{\delta=0} \phi_\delta(x,t) = V(x,t)$ and $d\pi_{(x,t)}V(x,t) = \sum_{j=1}^m t_j\, W_j(x)$ (see (2.3)) gives (2.5).

Set $E|Y = \pi^{-1}(Y)$. Condition (2.3) implies that the spray $s = \pi \circ \phi_1$ maps the domain $\Omega \cap E|Y$ to $Y$, so it is a local holomorphic spray on $Y$. Since the vector fields $W_1, \ldots, W_m$

generate the tangent space $T_xY$ every point $x \in Y_0 = Y \cap U_0$, we see from (2.5) that the restricted spray $s : \Omega \cap E|Y \to Y$ is dominating on $Y_0$. On the other hand, since $V$ vanishes on $E|\Lambda_0$, $\phi_1$ is the identity on this set and the spray $s = \pi$ is trivial over $\Lambda_0$.

In order to find a local dominating spray on $Y$, we proceed as follows. For $\alpha \in \{0, 1, \dots, n\}$, set $Y_\alpha = Y \cap U_\alpha$; this is an algebraic submanifold of $U_\alpha \cong \mathbb{C}^n$. Choose $m \in \mathbb{N}$ big enough that the tangent bundle $TY_\alpha$ is pointwise generated by $m$ polynomial vector fields $W_j^\alpha$ of the form (2.1) on $U_\alpha$ for every $\alpha \in \{0, 1, \dots, n\}$. In the affine coordinates $x = (z_0/z_\alpha, \dots, z_n/z_\alpha)$ on $U_\alpha$, we have $W_j^\alpha(x) = \sum_{i=1}^n V_{i,j}^\alpha(x)\partial_{x_i}$, where $V_{i,j}^\alpha$ are polynomials. Let

$$k_0 := \max_{\alpha,i,j} \deg V_{i,j}^\alpha + 2. \tag{2.7}$$

For $k \geq k_0$ the above argument gives an algebraic vector field $V^\alpha$ on the vector bundle $E^\alpha = m\mathbb{U}^k$ of the form (2.2) in the chart $E^\alpha|U_\alpha \cong U_\alpha \times \mathbb{C}^m$ that vanishes on $E_0^\alpha \cup E^\alpha|\Lambda_j$. Explicitly, if $t^\alpha = (t_1^\alpha, \dots, t_m^\alpha)$ are the fibre coordinates on $E^\alpha|U_\alpha$, then

$$V^\alpha(x, t^\alpha) = \sum_{i=1}^n \sum_{j=1}^m t_j^\alpha V_{i,j}^\alpha(x)\partial_{x_i}. \tag{2.8}$$

(We can take $V^0$ to be the vector field $V$ in (2.2), and every $V^\alpha$ is of the same form on $E^\alpha|U_\alpha$.) Let $E = E^0 \oplus E^1 \oplus \cdots \oplus E^n = (n+1)m\mathbb{U}^k$ and denote the vector bundle projection by $\pi : E \to \mathbb{CP}^n$. The algebraic vector field $V^\alpha$ on $E^\alpha$ can be extended to an algebraic vector field on $E$ by first extending it trivially (horizontally) to each of the summands $E^\beta|U_\alpha$ of $E|U_\alpha$ for $\beta \neq \alpha$ (note that these are trivial bundles), and then observing that the resulting vector field on $E|U_\alpha$ extends to an algebraic vector field on $E$ taking into account the condition (2.7) (see Lemma 2.1). With these extensions in place, we consider the vector field $V = \sum_{\alpha=0}^n V^\alpha$ on $E$. The construction implies that

$$d\pi_e V(e) \in T_yY \quad \text{for every } y \in Y \text{ and } e \in E_y = \pi^{-1}(y). \tag{2.9}$$

Since each $V^\alpha$ vanishes on the zero section of $E_0$ of $E$, so does $V$. Hence, there is a neighbourhood $\Omega \subset E$ of $E_0$ with convex fibres such that the flow $\phi_\tau(e)$ of $V$ exists for any initial point $e \in \Omega$ and every $\tau \in [0, 1]$. Consider the holomorphic spray

$$s = \pi \circ \phi_1 : \Omega \to \mathbb{CP}^n. \tag{2.10}$$

We claim that $s : \Omega \cap \pi^{-1}(Y) \to Y$ is dominating. To see this, consider the vector field $V = \sum_{\alpha=0}^n V^\alpha$ on a chart $E|U_\alpha$. For simplicity of notation we assume that $\alpha = 0$; the argument will be the same in every case. In the affine coordinates $x = (z_1/z_0, \dots, z_n/z_0)$ on $U_0$ and fibre coordinates $t = (t^0, t^1, \dots, t^n)$ on $E|U_0$, where $t^\alpha = (t_1^\alpha, \dots, t_m^\alpha)$ are fibre coordinates on the direct summand $E^\alpha|U_0$ of $E|U_0$, we have

$$V(x,t) = \sum_{\alpha=0}^n \sum_{i=1}^n \sum_{j=1}^m t_j^\alpha V_{i,j}^\alpha(x)\partial_{x_i} + \widetilde{V}(x,t) = \Theta(x,t) + \widetilde{V}(x,t) \tag{2.11}$$

where $|\widetilde{V}(x,t)| = O(|t|^2)$. That is, the vertical component $\widetilde{V}$ of $V$ vanishes to the second order along the zero section $E_0 = \{t = 0\}$. This follows from (2.4), which gives an explicit expression for a horizontal vector field of the form (2.2) in another affine vector bundle chart on $m\mathbb{U}^k$. Since the vector field $\Theta(x,t)$ in (2.11) is linear in the fibre variable $t$, the argument given above shows that its flow $\psi_\tau$ satisfies $\pi\circ\psi_\tau(x,\delta t)=\pi\circ\psi_{\delta\tau}(x,t)$ (cf. (2.6)). As before, it follows that the vertical derivative of the spray $\tilde s=\pi\circ\psi_1:\Omega\to\mathbb{CP}^n$ equals

$$(Vd\tilde s)_x(x,t)=\sum_{\alpha=0}^{n}\sum_{j=1}^{m} t_j^\alpha\, W_j^\alpha(x).$$

(Compare with (2.5).) Since the vectors $W_j^\alpha(x)$ span $T_xY$ for every $x\in Y_0$, the spray $\tilde s$ is dominating over $Y_0$. Since the second term $\widetilde V$ in (2.11) is of size $O(|t|^2)$, a standard argument using Grönwall's inequality shows that the flow $\phi_\tau$ of $V$ satisfies

$$\phi_\tau(x,t)=\psi_\tau(x,t)+O(|t|^2)\ \text{ as } |t|\to 0 \text{ and } \tau\in[0,1].$$

It follows that the spray $s=\pi\circ\phi_1:\Omega\to\mathbb{CP}^n$ (2.10) satisfies $(Vds)_x(x,t)=(Vd\tilde s)_x(x,t)$ for $x\in U_0$. The same argument holds on every chart $E|U_\alpha$, which proves that the spray $s:\Omega\cap\pi^{-1}(Y)\to Y$ is dominating as claimed.

So far, we have not used the hypothesis that $Y$ is an Oka manifold. We now replace the bundle $E\to\mathbb{CP}^n$ by its restriction $E|Y\to Y$, and we replace the spray $s$ (2.10) by its restriction $E|Y\cap\Omega\to Y$. Since $E$ is a direct sum of copies of the negative line bundle $\mathbb{U}^k|Y$, it is a 1-convex manifold with the exceptional subset $E_0\cong Y$ ($E_0$ is the maximal compact complex subvariety of $E$ without point components; see Grauert [15, Satz 1, p. 341]). Such $E$ admits a plurisubharmonic exhaustion function $\rho:E\to[0,\infty)$ which vanishes on $E_0$ and is positive and strongly plurisubharmonic on $E\setminus E_0$. In particular, $E_0$ admits a basis of strongly pseudoconvex neighbourhoods with convex fibres. As $Y$ is an Oka manifold, the results of Prezelj [26, 27] give a global holomorphic map $\tilde s:E\to Y$ which agrees with the spray $s$ to the second order along the zero section $E_0$ of $E$. Hence, $\tilde s$ is dominating, so $Y$ is elliptic. This completes the proof of Theorem 1.1.

Let us provide the details for the last step of the proof. We recall the special case of Prezelj's result which is needed. Assume that $X$ is a 1-convex manifold with the exceptional subvariety $Y$, $h:Z\to X$ is a holomorphic fibre bundle with an Oka fibre, $K$ is a compact holomorphically convex subset of $X$ containing $Y$, and $a:X\to Z$ is a continuous section of $h:Z\to X$ which is holomorphic on a neighbourhood of $K$. Then, the main result of [26] (whose proof is completed in [27]) gives a homotopy of continuous sections $a_t:X\to Z$, $t\in[0,1]$, such that $a_t$ agrees with $a$ to any given finite order along $Y$, the sections $a_t$ are holomorphic on a neighbourhood of $K$, they approximate $a$ as closely as desired on $K$, and the section $a_1$ is holomorphic on $X$.

In the case at hand, we apply Prezelj's result to the 1-convex manifold $X=E$ with the exceptional submanifold $E_0\cong Y$, letting $h:Z=X\times Y\to X$ be the projection with the Oka fibre $Y$, and $a:X\to Z$ be the graph of a continuous extension $a_0:X\to Y$ of the holomorphic map $s:U=\Omega\cap E\to Y$ constructed above. Such an extension clearly exists if we choose $U$ to have convex fibres and shrink $U$ if necessary. Choose the holomorphic section $a_1:X\to Z$ as in Prezelj's theorem, agreeing with $a_0$ to the second order along $Y$. The map $\tilde s=h\circ a_1:X=E\to Y$ is then a holomorphic spray on $Y$ with the required property.

**Remark 2.2.** In the proof of Theorem 1.1, we begin with suitably chosen algebraic (polynomial) vector fields on affine vector bundle charts, which extend to algebraic vector fields on a sufficiently negative vector bundle on the given manifold $Y$. This approach is only possible on projective manifolds since a compact complex manifold with a negative (or a positive) line bundle is necessarily projective. The subsequent techniques using flows of vector fields, and especially the last step involving Prezelj's theorem, are transcendental. Hence, this method does not give algebraic ellipticity, and it does not seem to give any hints as to how subellipticity could be shown to imply ellipticity in holomorphic settings.

## 3. Further results and remarks on ellipticity

In this section we collect some further observations concerning the relationship between the Oka property and ellipticity of a complex manifold.

**Remark 3.1.** If $L \to Y$ is a negative holomorphic line bundle on a compact (hence projective) manifold $Y$, then for sufficiently large $k > 0$, the vector bundle $\mathrm{Hom}(L^k, TY) \cong L^{-k} \otimes TY$ on $Y$ is generated by finitely many global holomorphic sections $h_1, \ldots, h_N$ (theorem of Hartshorne; see Lazarsfeld [25, Theorem 6.1.10]). Let $E = NL^k$ denote the direct sum of $N$ copies of $L^k$. Considering $h_i$ as a homomorphism $h_i : L^k \to TY$, it follows that the holomorphic vector bundle map $h = \oplus_{i=1}^N h_i : E \to TY$ is an epimorphism. Gromov proposed [16, 3.2.A', Step 2, p. 879] that such $h$ is the vertical derivative of a local dominating holomorphic spray $s : U \to Y$ from an open neighbourhood $U \subset E$ of its zero section $E_0 \cong Y$. This would give a shorter proof of Theorem 1.1. Although we do not know how to justify Gromov's claim, our proof of Theorem 1.1 follows this idea in spirit if not to the letter. This raises the following question.

**Problem 3.2.** Which holomorphic vector bundles $E \to Y$ of $\mathrm{rank}\, E \geq \dim Y$ admit a local dominating spray $s : U \to Y$ from a neighbourhood $U \subset E$ of the zero section of $E$?

The following observation generalises [12, Proposition 6.2]. Recall that every complex homogeneous manifold is elliptic [11, Proposition 5.6.1], hence an Oka manifold.

**Proposition 3.3.** *Assume that a compact complex manifold $Y$ admits a local dominating holomorphic spray $(E, \pi, s)$. If the bundle $\pi : E \to Y$ is generated by global holomorphic sections, then $Y$ is a complex homogeneous manifold.*

The condition on $E$ to be globally generated holds for a trivial bundle and for any sufficiently Griffiths positive bundle, but fails for negative bundles.

*Proof.* Let $s : U \to Y$ be a local dominating spray defined on a neighbourhood $U \subset E$ of the zero section $E_0$. The vertical derivative $Vds|E_0 : VT(E)|E_0 = E \to TY$ is a vector bundle epimorphism. Given a holomorphic section $\xi : Y \to E$, the map

$$Y \ni y \mapsto V_\xi(y) := Vds(y)(\xi(y)) \in T_yY$$

is a holomorphic vector field on $Y$. (We are using the natural identification of the vertical tangent bundle $VT(E)|E_0$ on the zero section $E_0$ with the bundle $E$ itself.) Applying this argument to sections $\xi_1, \ldots, \xi_m : Y \to E$ generating $E$ gives holomorphic vector fields $V_1, \ldots, V_m$ on $Y$ spanning the tangent bundle $TY$ since $Vds$ is surjective. Thus, $Y$ is holomorphically flexible. Since $Y$ is compact, these vector fields are complete, so their flows are complex 1-parameter subgroups of the holomorphic automorphism group $\mathrm{Aut}\, Y$, which is a finite-dimensional complex Lie group [5]. The spanning property implies that $\mathrm{Aut}\, Y$ acts transitively on $Y$. $\square$

There are projective Oka manifolds that are not homogeneous, for instance, blowups of certain projective manifolds such as projective spaces, Grassmannians, etc.; see [11, Propositions 6.4.5 and 6.4.6], the papers [17, 23], and the survey [6, Subsect. 6.3]. Many of these manifolds are algebraically elliptic. Another class of non-homogeneous projective surfaces that are algebraically elliptic are the Hirzebruch surfaces $H_l$ for $l = 1, 2, \ldots$; see [4, p. 191] and [11, Proposition 6.4.5]. In view of Proposition 3.3, such manifolds do not admit a local dominating spray from any globally generated holomorphic vector bundle.

**Remark 3.4.** Let $\mathscr{S}$ be the largest class of complex manifolds for which the Oka property implies ellipticity, that is, the class of manifolds that are either elliptic or not Oka. As remarked above, it is long known that every Stein manifold belongs to $\mathscr{S}$. By Theorem 1.1, so does every projective manifold. We know of two ways to produce new members of $\mathscr{S}$ from old. If $Y \to X$ is a covering map and $X$ is elliptic, so is $Y$. Also, $X$ is Oka if and only if $Y$ is. Hence, a covering space of a manifold in $\mathscr{S}$ is in $\mathscr{S}$. Also, it is easily seen that a product of manifolds in $\mathscr{S}$ is in $\mathscr{S}$.

**Acknowledgements.** Forstnerič is supported by the European Union (ERC Advanced grant HPDR, 101053085) and grants P1-0291 and N1-0237 from ARIS, Republic of Slovenia. The work on this paper was done during his visit to the University of Adelaide in February 2025, and he wishes to thank the institution for hospitality.

## References

[1] A. Alarcón, F. Forstnerič, and F. Lárusson. Isotopies of complete minimal surfaces of finite total curvature. 2024. `https://arxiv.org/abs/2406.04767`.

[2] R. B. Andrist, N. Shcherbina, and E. F. Wold. The Hartogs extension theorem for holomorphic vector bundles and sprays. *Ark. Mat.*, 54(2):299–319, 2016.

[3] I. Arzhantsev, S. Kaliman, and M. Zaidenberg. Varieties covered by affine spaces, uniformly rational varieties and their cones. *Adv. Math.*, 437:18, 2024. Id/No 109449.

[4] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge*. Springer-Verlag, Berlin, second edition, 2004.

[5] S. Bochner and D. Montgomery. Groups on analytic manifolds. *Ann. of Math. (2)*, 48:659–669, 1947.

[6] F. Forstnerič. Recent developments on Oka manifolds. *Indag. Math., New Ser.*, 34(2):367–417, 2023.

[7] F. Forstnerič. The Oka principle for sections of subelliptic submersions. *Math. Z.*, 241(3):527–551, 2002.

[8] F. Forstnerič. Holomorphic flexibility properties of complex manifolds. *Amer. J. Math.*, 128(1):239–270, 2006.

[9] F. Forstnerič. Runge approximation on convex sets implies the Oka property. *Ann. of Math. (2)*, 163(2):689–707, 2006.

[10] F. Forstnerič. Oka manifolds: from Oka to Stein and back. *Ann. Fac. Sci. Toulouse Math. (6)*, 22(4):747–809, 2013. With an appendix by Finnur Lárusson.

[11] F. Forstnerič. *Stein manifolds and holomorphic mappings (The homotopy principle in complex analysis)*, volume 56 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*. Springer, Cham, second edition, 2017.

[12] F. Forstnerič. Mergelyan's and Arakelian's theorems for manifold-valued maps. *Mosc. Math. J.*, 19(3):465–484, 2019.

[13] F. Forstnerič and F. Lárusson. Survey of Oka theory. *New York J. Math.*, 17A:11–38, 2011.

[14] F. Forstnerič and J. Prezelj. Oka's principle for holomorphic fiber bundles with sprays. *Math. Ann.*, 317(1):117–154, 2000.

[15] H. Grauert. Über Modifikationen und exzeptionelle analytische Mengen. *Math. Ann.*, 146:331–368, 1962.

[16] M. Gromov. Oka's principle for holomorphic sections of elliptic bundles. *J. Amer. Math. Soc.*, 2(4):851–897, 1989.

[17] S. Kaliman, F. Kutzschebauch, and T. T. Truong. On subelliptic manifolds. *Israel J. Math.*, 228(1):229–247, 2018.

[18] S. Kaliman and M. Zaidenberg. Algebraic Gromov ellipticity of cones over projective manifolds. *Math. Res. Lett.*, 31(6):1785–1817, 2024.

[19] S. Kaliman and M. Zaidenberg. Algebraic Gromov's ellipticity of cubic hypersurfaces. 2024. https://arxiv.org/abs/2402.04462.

[20] S. Kaliman and M. Zaidenberg. Gromov ellipticity and subellipticity. *Forum Math.*, 36(2):373–376, 2024.

[21] Y. Kusakabe. Elliptic characterization and localization of Oka manifolds. *Indiana Univ. Math. J.*, 70(3):1039–1054, 2021.

[22] Y. Kusakabe. Oka properties of complements of holomorphically convex sets. *Ann. Math. (2)*, 199(2):899–917, 2024.

[23] F. Lárusson and T. T. Truong. Algebraic subellipticity and dominability of blow-ups of affine spaces. *Doc. Math.*, 22:151–163, 2017.

[24] F. Lárusson and T. T. Truong. Approximation and interpolation of regular maps from affine varieties to algebraic manifolds. *Math. Scand.*, 125(2):199–209, 2019.

[25] R. Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 2004.

[26] J. Prezelj. A relative Oka-Grauert principle for holomorphic submersions over 1-convex spaces. *Trans. Amer. Math. Soc.*, 362(8):4213–4228, 2010.

[27] J. Prezelj. Positivity of metrics on conic neighborhoods of 1-convex submanifolds. *Internat. J. Math.*, 27(5):1650047, 24, 2016.

[28] J.-P. Serre. Géométrie algébrique et géométrie analytique. *Ann. Inst. Fourier, Grenoble*, 6:1–42, 1955–1956.

[29] M. Zaidenberg. Algebraic Gromov ellipticity: a brief survey. 2024. https://arxiv.org/abs/2409.04776.



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