OKA-1 MANIFOLDS: NEW EXAMPLES AND PROPERTIES

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ABSTRACT. In this paper we investigate Oka-1 manifolds and Oka-1 maps, a class of complex manifolds and holomorphic maps recently introduced by Alarcón and Forstnerič. Oka-1 manifolds are characterised by the property that holomorphic maps from any open Riemann surface to the manifold satisfy the Runge approximation and Weierstrass interpolation conditions, while Oka-1 maps enjoy similar properties for liftings of maps from open Riemann surfaces in the absence of topological obstructions. We also formulate and study the algebraic version of the Oka-1 condition, called aOka-1. We show that it is a birational invariant for compact algebraic manifolds and holds for all rational manifolds. This gives a Runge approximation theorem for maps from compact Riemann surfaces to uniformly rational projective manifolds. Finally, we study a class of complex manifolds with an approximation property for holomorphic sprays of discs. This class lies between the smaller class of Oka manifolds and the bigger class of Oka-1 manifolds and has interesting functorial properties.

1. INTRODUCTION

The class of Oka-1 manifolds was introduced in the literature by Alarcón and Forstnerič [2]. These are complex manifolds that admit plenty of holomorphic curves parametrised by an arbitrary open Riemann surface. Here is the precise definition (see [2, Definition 1.1]).

Definition 1.1. A complex manifold X is an *Oka-1 manifold* if for any open Riemann surface R, Runge compact set K in R, closed discrete set $A \subset R$, continuous map $f : R \to X$ which is holomorphic on a neighbourhood of $K \cup A$, number $\epsilon > 0$, and function $k : A \to \mathbb{N} = \{1, 2, \ldots\}$ there is a holomorphic map $F : R \to X$ which is homotopic to f and satisfies

(1) $\max_{p \in K} \operatorname{dist}_X(F(p), f(p)) < \epsilon$, and
(2) F agrees with f to order $k(a)$ at every point $a \in A$.

Here and in the sequel, dist_X denotes a distance function on the manifold X inducing the standard manifold topology. The properties under consideration will be independent of the particular choice of such a distance function.

Clearly, a complex manifold X is Oka-1 if and only if every connected component of X is such. By [2, Proposition 2.7], the conditions in Definition 1.1 are equivalent to the Runge approximation condition for any special pair of domains $K \subset K' = K \cup D \subset R$, where D is a closed disc attached to K along a boundary arc, with jet interpolation in finitely many points of K. Every Oka manifold (see [16, 19]) is also an Oka-1 manifold; the converse is false in general, at least for noncompact manifolds.
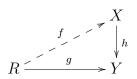
Here is a brief summary of the main results from [2]. We denote by $\mathcal{H}^p$ the p-dimensional Hausdorff measure on X with respect to a fixed Riemannian metric.

(a) If a complex manifold X of dimension n is dominable by $\mathbb{C}^n$ at every point in the complement of a closed subset $E \subset X$ with $\mathcal{H}^{2n-1}(E) = 0$ (such X is called *densely dominable* by $\mathbb{C}^n$), then X is an Oka-1 manifold (see [2, Theorem 2.2 and Corollary 2.5]).

(b) All Kummer surfaces, all elliptic K3 surfaces, and many elliptic surfaces of Kodaira dimension 1 are Oka-1 manifolds (see [2, Sect. 8]).

(c) Several functorial properties of Oka-1 manifolds are described in [2, Sect. 7].

Alarcón and Forstnerič also introduced the following notion of an *Oka-1 map*, by analogy with the notion of an Oka map (for the latter, see [33, Sect. 16] and [12, Definition 7.4.7]).

Definition 1.2 (Definition 7.7 in [2])**.** A holomorphic map $h : X \to Y$ of complex manifolds is an *Oka-1 map* if

(i) h is a Serre fibration, and

(ii) given an open Riemann surface R, a holomorphic map $g : R \to Y$, and a continuous lifting $f_0 : R \to X$ of g (that is, $h \circ f_0 = g$), which is holomorphic on a neighbourhood of a compact Runge subset $K \subset R$, we can deform f_0 through liftings of g to a holomorphic lifting $f : R \to X$ which approximates f_0 as closely as desired on K and agrees with f_0 to any given finite order at finitely many given points of K.

$$
\begin{array}{ccc}
 & & X \\
 & \nearrow^{f} & \downarrow h \\
R & \xrightarrow{\;\;g\;\;} & Y
\end{array}
$$

As noted in [2], a complex manifold X is an Oka-1 manifold if and only if the constant map $X \to point$ is an Oka-1 map. Every Oka map is also an Oka-1 map, but the converse fails, at least for maps with noncompact fibres. An Oka-1 map $X \to Y$ to a connected complex manifold Y is a surjective submersion and its fibres are Oka-1 manifolds [2, Proposition 7.8]. Oka-1 maps are useful in finding new examples of Oka-1 manifolds, as shown by the following results [2, Theorem 7.6 and Corollary 7.9]. (See [16, Theorem 3.15] for the analogous result for Oka manifolds and Oka maps.)

Theorem 1.3. *Let $h : X \to Y$ be an Oka-1 map between connected complex manifolds.*

(a) *If Y is an Oka-1 manifold, then X is an Oka-1 manifold.*

(b) *If X is an Oka-1 manifold and the homomorphism $h_* : \pi_1(X) \to \pi_1(Y)$ of fundamental groups is surjective, then Y is an Oka-1 manifold.*

(c) *If $h : X \to Y$ is a holomorphic fibre bundle with a connected Oka fibre, then X is an Oka-1 manifold if and only if Y is an Oka-1 manifold.*

Our first main result provides a large class of Oka-1 manifolds. The notion of *density property* is recalled in Sect. 2, and Theorem 2.2 establishes a stronger property of this class.

Theorem 1.4. *Let X be a Stein manifold with the density property and $\rho : X \to \mathbb{R}$ be a $\mathscr{C}^2$ Morse exhaustion function whose Levi form has at least two positive eigenvalues at each point. Then every superlevel set of ρ is an Oka-1 manifold.*

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By analogy with the algebraic version of the Oka property (see Lárusson and Truong [35]), it is natural to formulate and study the following algebraic version of the Oka-1 property.

Definition 1.5. A complex algebraic manifold X is *algebraically Oka-1* (abbreviated *aOka-1*) if for any affine algebraic Riemann surface R, Runge compact set K in R, finite set $A \subset K$, continuous map $f : R \to X$ which is holomorphic on a neighbourhood of K, number $\epsilon > 0$, and integer $k \in \mathbb{N}$ there is a morphism $F : R \to X$ which is homotopic to f and satisfies

(1) $\max_{p \in K} \mathrm{dist}_X(F(p), f(p)) < \epsilon$, and
(2) F agrees with f to order k at every point of A.

Recall that every affine algebraic Riemann surface R is the complement of finitely many points in a compact Riemann surface (and vice versa) and admits a regular embedding $R \hookrightarrow \mathbb{C}^3$ onto a closed smooth affine algebraic curve in $\mathbb{C}^3$.

Algebraic Oka properties for morphisms from affine algebraic varieties of arbitrary dimension to algebraic manifolds were introduced and studied by Lárusson and Truong [35]. They showed in particular that no algebraic manifold which is compact or contains a rational curve satisfies the algebraic version of the basic Oka property, the approximation property, or the interpolation property [35, Theorem 2]. The most useful known algebraic Oka property is the relative Oka principle for morphisms from affine algebraic varieties to algebraically subelliptic manifolds; see [16, Definition 6.1] for this notion, due to Gromov [22], and [11, Theorem 3.1] and the more precise version in [16, Theorem 6.4] for the relevant result. It was recently shown by Kaliman and Zaidenberg [24, Theorem 1.1] that every algebraically subelliptic manifold is in fact algebraically elliptic, and this property is equivalent to several other algebraic Oka properties. (See also [16, Theorem 6.2 and Corollary 6.6].)

The negative results of Lárusson and Truong, mentioned above, require sources of dimension at least two. Restricting to one-dimensional sources allows positive results to be proved. Our results in Sections 3 and 4 imply the following second main theorem of this paper. Its proof uses a multi-parameter version of the aforementioned homotopy approximation theorem [11, Theorem 3.1]; see Theorem 4.2.

Theorem 1.6. *A projective manifold that is birationally equivalent to an algebraically elliptic projective manifold is aOka-1.*

Recall that a connected complex manifold X is said to be *rationally connected* if any pair of points in X can be joined by a rational curve $\mathbb{P}^1 \to X$. Among many references for rationally connected projective manifolds, we refer to the papers by Kollár et al. [26, 28] and the monographs by Kollár [27] and Debarre [8]. It is shown in [28, Theorem 2.1] that several possible definitions of this class coincide. In particular, if every sufficiently general pair of points in X can be connected by an irreducible rational curve, or by a chain of such curves, then X is rationally connected. Also, if there is a morphism $f : \mathbb{P}^1 \to X$ such that $f^*TX \to \mathbb{P}^1$ is an ample bundle, then X is rationally connected. The class of projective rationally connected manifolds is birationally invariant. We have the following observation.

Proposition 1.7. *Every projective aOka-1 manifold is rationally connected.*

Proof. We may assume that the projective manifold X under consideration is connected. Take any pair of points $p, q \in X$. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. There is a holomorphic disc

$f_0 : \mathbb{D} \to X$ with $f_0(a) = p$ and $f_0(b) = q$ for some $a, b \in \mathbb{D}$. Assuming that X is aOka-1, there is a morphism $f : \mathbb{C} \to X$ with $f(a) = p$ and $f(b) = q$. Since X is projective, f extends to a morphism $f : \mathbb{P}^1 \to X$ with $p, q \in f(\mathbb{P}^1)$. This shows that X is rationally connected. $\quad\square$

Conversely, it is conjectured that every projective rationally connected manifold is Oka-1 [2, Conjecture 9.1]. As explained there, this would follow from a theorem of Gournay [21, Theorem 1.1.1], but the authors of [2] could not understand the details of the proof of this result. Gournay's theorem would also imply that every such manifold is an aOka-1 manifold. An Oka-1 type property of holomorphic maps $\mathbb{C} \to X$ to any rationally connected projective manifold X was established by Campana and Winkelmann [7], who constructed holomorphic lines $\mathbb{C} \to X$ with given jets through any given sequence of points in X. Their proof is based on the comb smoothing theorem by Kollár et al. [28].

In the present paper, we use different techniques to establish the aOka-1 property for two subclasses of the class of rationally connected projective manifolds: for rational manifolds (see Corollary 3.5), and for algebraically elliptic manifolds (see Theorem 4.1). Since the aOka-1 property is a birational invariant (see Corollary 3.4), these two results are summarised in Theorem 1.6. The theorem implies the following extension of Royden's Runge approximation theorem [36, Theorem 10] for maps from compact Riemann surfaces to the Riemann sphere $\mathbb{P}^1$ to a much wider class of targets; in particular, to any projective space $\mathbb{P}^n$. The corollary is proved in Section 4.

Corollary 1.8. *Let X be a projective manifold birationally equivalent to an algebraically elliptic projective manifold. Given a compact Riemann surface Σ, a compact subset K of Σ, a holomorphic map $f : U \to X$ from an open neighbourhood of K in Σ, and a finite subset A of K, there is a morphism $\Sigma \to X$ that approximates f arbitrarily closely on K and agrees with f to any given finite order in the points of A.*

There are many examples of projective Oka-1 manifolds that are not rationally connected, and hence not aOka-1 in view of Proposition 1.7. The simplest ones are elliptic curves. On the other hand, the following result is easily established.

Proposition 1.9. *Every aOka-1 manifold is Oka-1.*

Namely, by a result of Stout [38, Theorem 8.1], every compact smoothly bounded domain in an open Riemann surface is biholomorphic to a domain in an affine Riemann surface, so an inductive application of the aOka-1 condition yields the ostensibly weaker but equivalent formulation of the Oka-1 property in [2, Proposition 2.7] (see the paragraph following Definition 1.1 above).

Remark 1.10. By [6, Corollary 1.8], a compact algebraic manifold X of dimension n is unirational if and only if there is a surjective morphism $\mathbb{C}^n \to X$. Then X is densely dominable by $\mathbb{C}^n$, so X is Oka-1 by [2, Theorem 2.2]. (We thank an anonymous referee for pointing this out to us.) Whether X is in fact Oka or aOka-1 is unknown.

Finally, in Section 5 we continue the investigation of the local spray approximation property (LSAP) introduced in [2, Definition 7.11]. The Oka property clearly implies LSAP, which in turn implies the Oka-1 property by [2, Proposition 7.13]. We prove new functorial properties of the class LSAP (see Proposition 5.5) and deduce the following result.

Theorem 1.11. *Let $\mathscr{C}$ be the smallest class of complex manifolds that contains all Oka manifolds and is closed with respect to strong dominability. Then every manifold in $\mathscr{C}$ enjoys LSAP and hence is Oka-1.*

Saying that $\mathscr{C}$ is closed with respect to strong dominability means that if X is a complex manifold such that for every point $x \in X$ there is a manifold Y in $\mathscr{C}$ and a holomorphic map $f : Y \to X$ that is a submersion at some point in $f^{-1}(x)$, then X is in $\mathscr{C}$. It is an interesting open question whether the class of Oka manifolds is closed with respect to strong dominability (or merely Hausdorff-local). The question is open for the class of Oka-1 manifolds as well. The theorem shows that closing the class of Oka manifolds with respect to strong dominability does preserve some Oka-type properties: every manifold in the class thus obtained is Oka-1.

2. OKA-1 DOMAINS IN STEIN MANIFOLDS WITH THE DENSITY PROPERTY

The main result of this section (see Theorem 2.2) describes a large class of Oka-1 domains in Stein manifolds with the density property. It can be seen as an analogue of Kusakabe's theorem [31, Theorem 1.2] that the complement of any compact holomorphically convex set in a Stein manifold with the density property is an Oka manifold.

Recall that a holomorphic vector field on a complex manifold X is called *complete* if its flow exists for all complex values of time, so it forms a complex one-parameter group of holomorphic automorphisms of X. The following notion was introduced by Varolin [39]. (See also [12, Definition 4.10.1].)

Definition 2.1 (Varolin [39]). *A complex manifold X has the *density property* if every holomorphic vector field on X can be approximated uniformly on compacts by sums and commutators of complete holomorphic vector fields on X.*

This condition is most interesting and restrictive on Stein manifolds, since they admit many holomorphic vector fields. The fact that the Euclidean spaces $\mathbb{C}^n$ for $n > 1$ have the density property was discovered by Andersén and Lempert [3]. An important feature of Stein manifolds with the density property is that every isotopy of biholomorphic maps between pseudoconvex Runge domains in such a manifold can be approximated by an isotopy of holomorphic automorphisms (see Forstnerič and Rosay [20] and [12, Theorems 4.9.2 and 4.10.5]). It is known that most complex Lie groups and complex homogeneous manifolds of dimension greater than one have the density property, and there are numerous other classes of examples. Surveys of this subject can be found in [12, Sect. 4.10], [18], and [32].

The following is the main result of this section.

Theorem 2.2. *Let X be a Stein manifold with the density property and $\rho : X \to \mathbb{R}_+ = [0, \infty)$ be a $\mathscr{C}^2$ Morse exhaustion function whose Levi form has at least two positive eigenvalues at every point. Then, for every $c \in \mathbb{R}_+$, the domain*

$$X_c = \{x \in X : \rho(x) > c\}$$

is an Oka-1 manifold. Furthermore, given a compact Runge set K in an open Riemann surface R, a finite set $A \subset K$, and a continuous map $f : R \to X$ which is holomorphic on an open neighbourhood $U \subset R$ of K and satisfies $f(R \setminus \mathring{K}) \subset X_c$, there is a proper holomorphic map

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$F : R \to X$ which approximates f as closely as desired uniformly on K, agrees with f to a given finite order in the points of A, and satisfies $F(R \setminus \mathring{K}) \subset X_c$.

Proof. The first claim in the theorem, that X_c is an Oka-1 domain, follows from the second part of the theorem and [2, Proposition 2.7], which gives a criterion for a complex manifold to be Oka-1 by a simpler approximation condition and with interpolation in finitely many points.

We now prove the second part of the theorem. We may assume that R and X are connected. It clearly suffices to prove the result for a domain $X_{c_0} = \{\rho > c_0\}$ where $c_0 \in \mathbb{R}_+$ is a regular value of ρ. Let $A \subset K \subset U \subset R$ and $f : R \to X$ be as in the theorem with $c = c_0$. Pick a smoothly bounded Runge domain D in R with $K \subset D \subset \bar{D} \subset U$ such that

$$(2.1) \qquad\qquad f(\overline{D \setminus K}) \subset X_{c_0}.$$

Since $\rho : X \to \mathbb{R}_+$ is an exhaustion function and the manifold X is Stein, there exist a number $c_1 > c_0$ and a compact $\mathscr{O}(X)$-convex subset $L \subset X$ whose complement $O = X \setminus L$ satisfies

$$(2.2) \qquad\qquad X_{c_1} \subset O \subset X_{c_0}.$$

Since $\rho : X \to \mathbb{R}_+$ is a Morse exhaustion function whose Levi form has at least two positive eigenvalues at every point, [9, Theorem 1.1] and its proof (see in particular [9, Lemma 6.3]) furnishes a holomorphic map $\tilde{f} : \bar{D} \to X$ satisfying the following conditions for any given pair of numbers $\epsilon > 0$ and $k = 1, 2, \ldots$.

(a) $\tilde{f}(bD) \subset X_{c_1}$.
(b) $\tilde{f}(\overline{D \setminus K}) \subset X_{c_0}$.
(c) $\sup_{p \in K} \mathrm{dist}_X(\tilde{f}(p), f(p)) < \epsilon$.
(d) $\tilde{f}$ agrees with f to order k in every point of A.
(e) $\tilde{f}$ is homotopic to $f|_{\bar{D}}$ by a homotopy of maps $f_t : \bar{D} \to X$ with $f_0 = f|_{\bar{D}}$, $f_1 = \tilde{f}$ satisfying $f_t(\overline{D \setminus K}) \subset X_{c_0}$ for every $t \in [0, 1]$.

The assumption on the Levi form of ρ implies that it has Morse index at most $2 \dim_{\mathbb{C}} X - 2$ at every critical point. Hence, for any pair of regular values $c < c'$ of ρ the set $X_c = \{\rho > c\}$ is topologically obtained from $X_{c'}$ by attaching handles of dimension at least 2. It follows that the inclusion $X_{c'} \hookrightarrow X_c$ induces a bijection of the path components and the relative fundamental group $\pi_1(X_c, X_{c'})$ vanishes. Since the pair (R, D) is homotopy equivalent to a relative 1-dimensional CW complex, we infer that $\tilde{f}$ extends from $\bar{D}$ to a continuous map $\tilde{f} : R \to X$ satisfying the following two additional conditions:

(f) $\tilde{f}(R \setminus D) \subset X_{c_1}$.

(g) The homotopy $f_t : \bar{D} \to X$ in condition (e) extends to a homotopy $f_t : R \to X$ ($t \in [0, 1]$) satisfying $f_0 = f$, $f_1 = \tilde{f}$, and $f_t(R \setminus D) \subset X_{c_0}$ for every $t \in [0, 1]$.

Since the Stein manifold X is assumed to have the density property, the domain $O = X \setminus L$ in (2.2) is Oka by Kusakabe [31, Theorem 1.2]. From the condition $\tilde{f}(R \setminus D) \subset X_{c_1} \subset O$ (see (2.2) and (f)) and [16, Theorem 1.3] it follows that there is a holomorphic map $F : R \to X$ which approximates $\tilde{f}$ as closely as desired on $\bar{D}$, it agrees with $\tilde{f}$ (and hence with f) to order k in the points of A, it satisfies $F(R \setminus D) \subset O$, and F is homotopic to f through maps $R \to X$ sending $R \setminus \mathring{K}$ to X_0. Thus, F satisfies conditions (1) and (2) in Definition 1.1. In particular,

if $f(K) \subset X_{c_0}$ then the map F, and the homotopy from f to F, can be chosen to have range in X_{c_0}. By [2, Proposition 2.7] this shows that X_{c_0} is an Oka-1 manifold.

Furthermore, we can choose the map $F : R \to X$ as above to be a proper holomorphic immersion (embedding if $\dim X \geq 3$) provided the jet interpolation conditions allow it. This follows from the main result of Andrist and Wold [5]. (See also [4, 13] for an extension of their result to maps from any Stein manifold R satisfying $2 \dim R \leq \dim X$.) By using the recently established fact that $X \setminus L$ is an Oka domain when X is a Stein manifold with the density property and L is a compact $\mathscr{O}(X)$-convex subset of X (see Kusakabe [31, Theorem 1.2]), a simpler proof is possible; cf. [17, Theorem 5.1]. The main idea is the following. We exhaust R by an increasing sequence of compact $\mathscr{O}(R)$-convex subsets $K = K_0 \subset K_1 \subset \cdots \subset \bigcup_{i=0}^{\infty} K_i = R$. Likewise, we exhaust X by compact $\mathscr{O}(X)$-convex subsets $L = L_1 \subset L_2 \subset \cdots \subset \bigcup_{i=1}^{\infty} L_i = X$, where L is as above. (The sets L_i are chosen as sublevel sets of a strongly plurisubharmonic Morse exhaustion function $\tau : X \to \mathbb{R}_+$.) We then inductively construct a sequence of continuous maps $f_i : R \to X$ ($i = 0, 1, \ldots$) such that f_0 is the initial map, and for every $i = 1, 2, \ldots$ the map f_i is holomorphic on K_i, it approximates f_{i-1} as closely as desired on K_{i-1} and agrees with it to order k in the points of $A \subset K$, and it satisfies $f_i(R \setminus \mathring{K}_i) \subset O_i = X \setminus L_i$. To obtain such a sequence, we alternately use [9, Theorem 1.1] and [16, Theorem 1.3], just like in the above construction of the map $\tilde{f} = f_1$ from the given initial map $f = f_0$. If the approximations are close enough then there is a limit holomorphic map $F = \lim_{i \to \infty} f_i : R \to X$, which is proper as a map to X and satisfies the conditions in the theorem. $\qquad \square$

A characteristic feature of domains X_c in Theorem 2.2 is that, taking $c \in \mathbb{R}_+$ to be a regular value of ρ, the compact set

$$L_c = \{x \in X : \rho(x) \leq c\} = X \setminus X_c$$

is such that the Levi form of its boundary $bL_c = \{\rho = c\}$ has at least one positive eigenvalue at every point (since the Levi form of ρ has at least two positive eigenvalues at every point). Let us call a compact set $L \subset X$ with $\mathscr{C}^2$ boundary satisfying this property *tangentially 1-convex*. Note that every such set L is strongly pseudoconvex if $\dim X = 2$, but this fails if $\dim X \geq 3$. A specific example is $L = \{\rho \leq c\} \subset \mathbb{C}^3$ where $\rho(z_1, z_2, z_3) = |z_1|^2 + |z_2|^2 - |z_3|^2 + |z_3|^4$ and $c > 0$ is a regular value of ρ. Indeed, ρ is an exhaustion function on $\mathbb{C}^3$ whose Levi form has eigenvalues $1, 1, -1 + 4|z_3|^2$ at each point, so L is tangentially 1-convex. However, the signature of the Levi form of bL along the circle $\{|z_1|^2 = c, \ z_2 = z_3 = 0\} \subset bL$ is $(1, -1)$, so L is not strongly pseudoconvex.

Problem 2.3. Assume that X is a Stein manifold with the density property and $L \subset X$ is a compact tangentially 1-convex set. Is $O = X \setminus L$ and Oka-1 manifold?

Another reason for considering this question is the following. Every connected Oka-1 manifold is $\mathbb{C}$-connected and hence Liouville, that is, it does not admit any nonconstant bounded plurisubharmonic functions. If $L \subset X$ is a compact set whose $\mathscr{C}^2$ boundary contains a strongly pseudoconcave point $p \in bL$, then its complement $O = X \setminus L$ admits a local negative strongly plurisubharmonic peak function vanishing at p. The maximum of this function and a suitably chosen negative constant is a nonconstant negative plurisubharmonic function on O, so O is not Oka-1. On the other hand, pseudoconcave boundary points of L

are the only obstruction to $O = X \setminus L$ being Liouville. For simplicity, we consider the case $X = \mathbb{C}^n$, noting that the same argument applies in general. Given $\epsilon > 0$, we set

$$L_\epsilon = \big\{ z \in \mathbb{C}^n : \mathrm{dist}(z, L) = \min_{w \in L} |z - w| \leq \epsilon \big\}.$$

Proposition 2.4. *Let L be a compact subset of $\mathbb{C}^n$, $n > 1$. Assume that there are numbers $0 < \delta < \epsilon$ such that for every point $p \in L_\delta \setminus L$ there is a compact complex curve A in $\mathbb{C}^n \setminus L$ with $p \in A$ and $bA \subset \mathbb{C}^n \setminus L_\epsilon$. Then $\mathbb{C}^n \setminus L$ is Liouville.*

Proof. By the maximum principle, the condition in the proposition fails on any bounded connected component of $O = \mathbb{C}^n \setminus L$, and hence O is connected. Assume that ϕ is a bounded plurisubharmonic function on O. Then ϕ is constant on the Oka domain $\mathbb{C}^n \setminus \widehat{L} \subset O$, where $\widehat{L}$ denotes the polynomial hull of L. Let $p \in A \subset O$ be a complex curve as in the proposition. By the maximum principle for the subharmonic function $\phi|_A$ we have that $\phi(p) \leq \max_{bA} \phi$. Since $bA \subset \mathbb{C}^n \setminus L_\epsilon$ and this holds for every $p \in L_\delta \setminus L$, and since ϕ is constant on $\mathbb{C}^n \setminus \widehat{L}$, ϕ assumes its maximum value on O on the compact set $\widehat{L} \setminus \mathring{L}_\epsilon$. By the maximum principle for plurisubharmonic functions it follows that ϕ is constant on O. $\square$

Corollary 2.5. *If L is a compact tangentially 1-convex set in $\mathbb{C}^n$, then $\mathbb{C}^n \setminus L$ is Liouville.*

Proof. The existence of a positive Levi eigenvalue at every point $p \in bL$ gives an embedded holomorphic disc $\Delta_p \subset \mathbb{C}^n$ such that $\Delta_p \cap L = \{p\}$ and Δ_p is tangent to bL precisely to the second order. By compactness of bL we can find a family of such discs with boundaries contained in $\mathbb{C}^n \setminus L_\epsilon$ for some $\epsilon > 0$. Translating these discs aways from L for a small amount gives a family of holomorphic discs satisfying the conditions of Proposition 2.4, so the conclusion follows. $\square$

3. Oka-1 is a bimeromorphic invariant

It was a long-standing open problem whether the Oka property is a bimeromorphic invariant, until Kusakabe proved that there are non-tame discrete subsets A of $\mathbb{C}^n$, $n \geq 2$, for which $\mathbb{C}^n$ blown up at each point of A is Brody volume-hyperbolic and hence not Oka [30, Example A.3]. Previously, it was shown that if A is tame, then the blow-up is weakly subelliptic and hence Oka [12, Proposition 6.4.12]. It remains an open question whether an Oka manifold blown up at a single point is Oka.

On the algebraic side more is known. Kusakabe proved, by a reduction to [34, Theorem 1], that if an algebraic manifold X satisfies the algebraic convex approximation property, aCAP, then so does the blow-up B of X along any algebraic submanifold (not necessarily connected); see [29, Corollary 4.3]. It follows that if X is algebraically elliptic (see Section 4 for the definition and more information on this class), then B is Oka. The optimal known geometric sufficient condition for a compact algebraic manifold to be algebraically elliptic is uniform rationality (see Arzhantsev et al. [6, Theorem 1.3]). A compact algebraic manifold X is called uniformly rational if every point of X admits a Zariski open neighbourhood isomorphic to a Zariski open set in $\mathbb{C}^n$ with $n = \dim X$. If X is uniformly rational, so is the blow-up B of X along any algebraic submanifold (see Gromov [22, 3.5E]). Hence, if X is compact and uniformly rational, then B is algebraically elliptic. If X is of class $\mathcal{A}$ (see [12, Definition 6.4.4]), not necessarily compact, then B is algebraically elliptic [34, Corollary 2].

Further, algebraic ellipticity implies strong dominability by affine spaces [34, Proposition 6], and if X is dominable or strongly dominable, then so is B [34, Theorem 9]. In view of all these results, it is reasonable to expect that an algebraically elliptic manifold blown up along an algebraic submanifold is algebraically elliptic, but this is yet to be proved.

For 1-dimensional sources, the picture is much clearer.

Theorem 3.1. *Let B be the blow-up of a complex manifold X along a submanifold, not necessarily connected. Then B is Oka-1 if and only if X is Oka-1.*

By the solution of the weak factorisation conjecture due to Abramovich et al. [1, Theorem 0.3.1], a bimeromorphic map between compact complex manifolds can be factored into a finite sequence of blow-ups and blow-downs with smooth centres. Thus the following corollary to Theorem 3.1 is immediate.

Corollary 3.2. *For compact complex manifolds, the Oka-1 property is bimeromorphically invariant.*

Proof of Theorem 3.1. Let $\pi : B \to X$ be the projection, the submanifold Z of X be the centre of the blow-up, and $E = \pi^{-1}(Z)$ be the exceptional divisor in B. We may assume that $\operatorname{codim} Z \geq 2$. Let R be an open Riemann surface, $K \subset R$ be a Runge compact, and $A \subset R$ be discrete with a map $k : A \to \mathbb{N}$. Assuming that X is Oka-1, we take a continuous map $f : R \to B$ which is holomorphic on a neighbourhood of $K \cup A$. We wish to deform f to a holomorphic map $R \to B$ which approximates f on K and agrees with f to order $k(a)$ for each $a \in A$.

In preparation, we first add a point $b \in R$ to A away from K, set $k(b) = 1$, and deform f near b so that f is holomorphic near b and $f(b) \notin E$. Denote the new A and new f by the same letters. This is only to make sure that when we deform $\pi \circ f$ below, we do not end up with a map into Z. Also, by the transversality theorem (see [23] or [12, Theorem 8.8.13]), after applying an arbitrarily small deformation to f, we may assume that no connected component of the neighbourhood of K on which f is holomorphic maps into E. Then there are only finitely many points $x \in K \setminus A$ with $f(x) \in E$. We add every such x to A and set $k(x) = 1$. Finally, for every $a \in A \cap f^{-1}(E)$, we add 1 to $k(a)$.

We now apply the Oka-1 property of X to the map $g = g_0 = \pi \circ f$ and obtain a homotopy $t \mapsto g_t : R \to X$, such that g_1 is holomorphic, approximates g on K, and agrees with g to order $k(a)$ at each $a \in A$. The preparations above ensure that g_1 has a unique holomorphic lifting $f_1 : R \to B$ by $\pi : B \to X$ with the required approximation and interpolation properties. It remains to verify that f_1 is homotopic to f. Since R has the homotopy type of a bouquet of circles and deformation-retracts onto an embedded bouquet, the homotopy classes of continuous maps from R to any space Y are in bijective correspondence with homomorphisms $\pi_1(R) \to \pi_1(Y)$ of their fundamental groups. As $\pi : B \to X$ induces an isomorphism $\pi_1(B) \to \pi_1(X)$ (this is easily shown using the van Kampen theorem), we conclude that f and f_1 are homotopic.

Next, assuming that B is Oka-1, we take a continuous map $g : R \to X$ which is holomorphic on a neighbourhood of $K \cup A$. We wish to deform g to a holomorphic map $R \to X$ which approximates g on K and agrees with g to order $k(a)$ for each $a \in A$. Once we are able to lift g to a continuous map $f : R \to B$, which is holomorphic on a neighbourhood

of $K \cup A$, we simply apply the Oka-1 property of B to obtain a suitable deformation of f and then compose with π. Since the submanifold Z of X has codimension at least $2 > \dim R$, by applying an arbitrarily small deformation to g fixing every point $a \in A$ to order $k(a)$, we may assume that $g^{-1}(Z) \subset A$, and then the existence of a lifting f is clear. $\qquad \square$

In the algebraic setting, the analogous results are proved in exactly the same way, using the algebraic version of the universal property of the blow-up and the algebraic weak factorisation theorem of Abramovich et al. [1, Theorem 0.1.1]. This gives the following conclusions.

Theorem 3.3. *Let B be the blow-up of an algebraic manifold X along an algebraic submanifold, not necessarily connected. Then B is aOka-1 if and only if X is aOka-1.*

Corollary 3.4. *For compact algebraic manifolds, the algebraic Oka-1 property is birationally invariant.*

The corollary, combined with Royden's Runge approximation theorem for maps from Riemann surfaces to $\mathbb{P}^1$ [36, Theorem 10], shows the following.

Corollary 3.5. *Every compact rational manifold is aOka-1. In particular, the projective space $\mathbb{P}^n$ is aOka-1.*

Proof. By Royden's theorem [36, Theorem 10], the Riemann sphere $\mathbb{P}^1$ is aOka-1. The required approximation and jet interpolation is precisely the content of his theorem. The required homotopy is then immediate, because an open Riemann surface has the homotopy type of a bouquet of circles and $\mathbb{P}^1$ is simply connected. Applying this result componentwise, it follows that every product $(\mathbb{P}^1)^n$ of Riemann spheres is aOka-1. Since $(\mathbb{P}^1)^n$ is a rational manifold, Corollary 3.4 concludes the proof. $\qquad \square$

Corollary 3.5 complements Theorem 4.1 in the following section, which shows that every algebraically elliptic projective manifold is aOka-1. It is unknown whether every rational manifold is Oka, let alone algebraically elliptic.

4. SIMPLY CONNECTED ALGEBRAICALLY ELLIPTIC MANIFOLDS ARE AOKA-1

In this section, we prove the following main result.

Theorem 4.1. *Every simply connected algebraically elliptic manifold is aOka-1. In particular, every algebraically elliptic projective manifold is aOka-1.*

Algebraically elliptic manifolds have already been mentioned in the introduction and in Section 3. An algebraic manifold X, not necessarily compact or quasi-projective, is said to be algebraically elliptic if it admits a dominating algebraic spray $E \to X$ defined on the total space of an algebraic vector bundle $E \to X$; see [16, Definition 6.1 (a)]. This notion was introduced and studied by Gromov [22]. The ostensibly bigger class of algebraically subelliptic manifolds (see [16, Definition 6.1 (b)]) was implicitly present in Gromov's work but was formally introduced in [10]. Recently it was shown by Kaliman and Zaidenberg [24, Theorem 1.1] that these two classes coincide. However, it is not known whether the same is true in the holomorphic category. It was recently shown by Arzhantsev et al. that every compact uniformly rational manifold is algebraically elliptic [6, Theorem 1.3]. Note that an

algebraically elliptic projective manifold is unirational, and hence simply connected by Serre [37]. This justifies the second statement in Theorem 4.1 provided that the first statement holds, and it complements Corollary 3.5, which says that a compact rational manifold is aOka-1.

Rational manifolds and algebraically elliptic manifolds form two subclasses of the class of unirational manifolds. It has recently been shown by Kaliman and Zaidenberg [25] that every smooth cubic hypersurface in $\mathbb{P}^n$, $n > 2$, is algebraically elliptic. This seems to give the first examples of non-rational projective manifolds which are algebraically elliptic. On the other hand, it is unknown whether every rational manifold is algebraically elliptic.

Algebraic ellipticity is a Zariski local property, and is equivalent to several other algebraic Oka properties; see the summary in [16, Theorem 6.2 and Corollary 6.6]. One is the *algebraic homotopy approximation property* [16, Theorem 6.4] whose basic version was proved by Forstnerič [11, Theorem 3.1]. We shall need the following multi-parameter version of this result. We consider the cube $[0,1]^n \subset \mathbb{R}^n \subset \mathbb{C}^n$ as a subset of $\mathbb{C}^n$ in a natural way.

Theorem 4.2. *Let Σ be an affine algebraic variety and X be an algebraically elliptic manifold. Given a morphism $f : \Sigma \to X$, a compact holomorphically convex set K in Σ, an open neighbourhood $U \subset \Sigma$ of K, and a homotopy of holomorphic maps*

$$(4.1) \qquad\qquad f_t : U \to X, \quad t = (t_1, t_2, \ldots, t_n) \in [0,1]^n$$

with $f_0 = f|_U$, there are morphisms $F : \Sigma \times \mathbb{C}^n \to X$ with $F(\cdot, 0) = f$ such that $F(\cdot, t)$ approximates f_t as closely as desired uniformly on $K \times [0,1]^n$.

If in addition the homotopy f_t is fixed on a closed algebraic subvariety $\Sigma' \subset \Sigma$ then F can be chosen such that $F(x,t) = f(x)$ for all $x \in \Sigma'$ and $t \in \mathbb{C}^n$.

Proof. The case $n = 1$ coincides with [16, Theorem 6.4]. We proceed by induction on n. Assume that $n > 1$ and write the parameter in the form $t = (t_1, t')$, with $t_1 \in [0,1]$ and $t' \in [0,1]^{n-1}$. Choose a compact $\mathscr{O}(\Sigma)$-convex set $K_1 \subset U$ containing K in its interior. Applying the result for $n = 1$ to K_1 and the parameter t_1 (with $t' = 0$) gives a morphism $F_1 : \Sigma \times \mathbb{C} \to X$ such that $F_1(\cdot, 0) = f$ and $F_1(\cdot, t_1)$ approximates $f_{(t_1, 0')}$ uniformly on $K_1 \times [0,1]$. We now adjust the homotopy f_t in (4.1) (without changing the notation) such that $f_{(t_1, 0')} = F_1(\cdot, t_1)$ for all $t_1 \in [0,1]$ and the new homotopy is uniformly close to the original one on $K_1 \times [0,1]^n$. This can be done by using the partition of unity in the parameter and the existence of open Stein neighbourhoods of graphs of the maps f_t over K_1; see e.g. [12, proof of Proposition 6.7.2], specifically the last paragraph on p. 289.

We now replace Σ by the affine variety $\Sigma_1 = \Sigma \times \mathbb{C}$, the morphism f by F_1, and the homotopy f_t by the $(n-1)$-parameter homotopy $f_{(\cdot, t_2, \ldots, t_n)}$ with $(t_2, \ldots, t_n) \in [0,1]^{n-1}$. Applying the same argument as above with the parameter $t_2 \in [0,1]$ and (if $n > 2$) with $t_3 = \cdots = t_n = 0$ yields a morphism $F_2 : \Sigma_2 = \Sigma \times \mathbb{C}^2 \to X$ satisfying the required properties with respect to the homotopy $f_{(t_1, t_2, 0')}$. Clearly this process can be continued inductively, and in n steps we obtain the desired morphism $F = F_n : \Sigma \times \mathbb{C}^n \to X$ satisfying the conclusion of the theorem.

As pointed out in [16, Theorem 6.4], a minor modification of this argument also gives the addition with interpolation on an algebraic subvariety $\Sigma' \subset \Sigma$ on which the original homotopy f_t is fixed. (We shall not need this addition in the present paper.) $\qquad\square$

Remark 4.3. The hypothesis in Theorem 4.2 that the member f_0 of the family $\{f_t\}_{t\in[0,1]^n}$ is the restriction of a morphism $\Sigma \to X$ can be replaced by the hypothesis that some (and hence any) member of the family $\{f_t\}_{t\in[0,1]^n}$ is homotopic to the restriction $g|_U$ of a morphism $g : \Sigma \to X$. Since X is an Oka manifold (see [12, Corollary 5.6.14]) and assuming as we may that U is Stein, the Oka principle gives a homotopy of holomorphic maps $U \to X$ connecting g to the map f_0, and hence to any map f_t in the family. The result then follows by rearranging the set of parameters of the extended homotopy into the cube $[0, 1]^{n+1}$ so that $g|_U$ corresponds to the point $(0, \ldots, 0)$ and f_t corresponds to the point $(1, t) \in \{1\} \times [0, 1]^n$.

Proof of Theorem 4.1. Let R be an affine algebraic Riemann surface, K be a Runge compact set in R, $A = \{a_1, \ldots, a_m\} \subset K$ be a finite set, and $f : R \to X$ be a continuous map which is holomorphic on a neighbourhood $U \subset R$ of K (see Definition 1.5). Our goal is to show that we can approximate f on K by morphisms $\tilde{f} : R \to X$ which agree with f to a given finite order k at every point of A. By enlarging K slightly, we may assume that $A \subset \mathring{K}$.

Let s denote the real dimension of the space of holomorphic k-jets at the origin of maps from open neigbourhood of $0 \in \mathbb{C}$ to X sending 0 to a chosen point $x_0 \in X$. Set $n = m(\dim_\mathbb{R} X + s)$. Note that the graph of f over K has an open Stein neighbourhood in $R \times X$ (indeed, it has an open Euclidean neighbourhood, see [14]). Hence, after shrinking U around K if necessary, we can find a smooth n-parameter family of holomorphic maps $f_t : U \to X$ for $t \in Q = [-1, 1]^n \subset \mathbb{R}^n$ such that $f_0 = f|_U$ and for every collection of values and k-jets near the values and k-jets of f_0 at the points $a_j \in A$ $(j = 1, \ldots, m)$ there is precisely one member f_t of this family which assumes these values and k-jets at the points of A. Thus, $\{f_t\}$ is a universal local deformation family of f_0 with respect to the values and k-jets at the points of A.

Since X is simply connected and U has the homotopy type of a bouquet of circles, f_0 is homotopic to the restriction $g|_U$ of a constant morphism $g : R \to x_0 \in X$. By Theorem 4.2 and Remark 4.3, we can approximate the homotopy $\{f_t\}$ for $t \in Q = [-1, 1]^n \subset \mathbb{C}^n$ as closely as desired on $K \times Q$ by a morphism $\Theta : R \times \mathbb{C}^n \to X$. Consider the morphisms $\Theta_t = \Theta(\cdot, t) : R \to X$ for $t \in Q$. By approximation, the values and k-jets of Θ_t in the points of A are arbitrarily close to those of f_t for all $t \in Q$. Fix a number $0 < r < 1$. Assuming as we may that the approximation is close enough, we obtain a continuous map $Q_r := [-r, r]^n \ni t \mapsto \tau(t) \in Q$ close to the identity map such that the values and k-jets of Θ_t at the points of A match those of $f_{\tau(t)}$, the set $\tau(bQ_r) \subset Q$ contains the center point 0 in the bounded component of its complement, and the map $\tau/|\tau| : bQ_r \to S^{n-1}$ to the $(n-1)$-sphere has topological degree 1. It follows that there is $t \in (-r, r)^n = \mathring{Q}_r$ such that $\tau(t) = 0$ (see [15, Proposition 3.2] and the references therein). For this t, the values and k-jets of the morphism $\tilde{f} = \Theta_t : R \to X$ in the points of A match those of f_0. Since $r > 0$ was arbitrary, $\tilde{f}$ may be chosen as close to f as desired on K. This completes the proof. $\qquad\square$

Proof of Corollary 1.8. We may assume that K is a proper subset of Σ, for otherwise there is nothing to prove. By enlarging K within U we may assume it is a smoothly bounded. Hence, $\Sigma \setminus K$ has finitely many connected components. Choose a finite set $P \subset \Sigma \setminus K$ having a point in each component of $\Sigma \setminus K$. Then, $R = \Sigma \setminus P$ is an affine Riemann surface and K is $\mathscr{O}(R)$-convex. Since X is simply connected, the map f extends to continuous map $f : R \to X$. By Theorem 4.1 and Corollary 3.4, X enjoys the aOka-1 property, which implies the existence

of a morphism $\tilde{f} : R \to X$ satisfying the approximation and interpolation conditions in the corollary. Since X is compact, $\tilde{f}$ extends to a morphism $\Sigma \to X$. $\qquad\square$

5. CONDITION LSAP AND ITS FUNCTORIAL PROPERTIES

In this section, we study a property of a complex manifold, called the *local spray approximation property* (abbreviated LSAP), which was introduced by Alarcón and Forstnerič [2, Definition 7.11]. This property implies that the manifold is Oka-1. We prove some new functorial properties of the class of LSAP manifolds (see Proposition 5.5 and Corollary 5.6), thereby extending the class of Oka-1 manifolds.

Definition 5.1. A pair of compact topological discs $D \subset D' \subset \mathbb{C}$ is a *special pair* if both discs have piecewise smooth boundaries and $D' \setminus \mathring{D}$ is a disc attached to D along an arc in bD.

A holomorphic map $F : D \times \mathbb{B}^N \to X$ to a complex manifold X is called a *holomorphic spray* of maps $D \to X$ with the core $f = F(\cdot, 0)$. Here, $\mathbb{B}^N$ is the open unit ball in $\mathbb{C}^N$. Such a spray is said to be *dominating* if the partial derivative $\dfrac{\partial}{\partial t}\bigg|_{t=0} F(z, t) : \mathbb{C}^N \to T_{f(z)}X$ is surjective for all $z \in D$.

Definition 5.2 (Definition 7.11 in [2])**.** A complex manifold X has the *local spray approximation property*, LSAP, at a point $x \in X$ if there is a neighbourhood $V \subset X$ of x satisfying the following condition. Given a special pair of compact discs $D \subset D' \subset \mathbb{C}$ and a holomorphic spray $F : D \times \mathbb{B}^N \to V$, there is a number $r = r(F) \in (0, 1)$ such that F can be approximated uniformly on $D \times r\overline{\mathbb{B}}^N$ by holomorphic sprays $D' \times r\overline{\mathbb{B}}^N \to X$. The manifold X has LSAP, or is an LSAP manifold, if this holds for every $x \in X$.

Remark 5.3. The sprays in Definition 5.2 can either be defined over unspecified open neighbourhoods of the compact disc D, or else they could be continuous over D and holomorphic over its interior $\mathring{D}$. The application of this condition in the proofs of Propositions 5.4 and 5.5 works in both cases, since the technique of gluing sprays on Cartan pairs works in both cases (see [12, Sect. 5.9]). As noted in [2, Remark 7.12], LSAP is equivalent to the ostensibly weaker condition that every *dominating* holomorphic spray $F : D \times \mathbb{B}^N \to V$ can be approximated by sprays on a given bigger disc $D' \supset D$ as in Definition 5.2.

It is obvious that $\mathbb{C}^n$ is an LSAP manifold. We recall the following results concerning the relationship between LSAP and Oka-1; see [2, Proposition 7.13].

Proposition 5.4. (a) *If a complex manifold X satisfies LSAP at every point $x \in X \setminus E$ in the complement of a closed subset $E \subset X$ with $\mathcal{H}^{2\dim X - 1}(E) = 0$, then X is an Oka-1 manifold. In particular, every LSAP manifold is an Oka-1 manifold.*
(b) *A complex manifold X which is densely dominable by LSAP manifolds is Oka-1.*
(c) *In particular, if $f : X \to Y$ is a surjective holomorphic submersion and X is an LSAP manifold, then Y is an Oka-1 manifold.*
(d) *A holomorphic fibre bundle $X \to Y$ with an LSAP fibre is an Oka-1 map.*

The condition in part (b) means that there is a closed subset $E \subset X$ satisfying the condition $\mathcal{H}^{2\dim X - 1}(E) = 0$ in part (a) such that for every point $x \in X \setminus E$, there exist an LSAP

manifold Y (possibly depending on x), a holomorphic map $h : Y \to X$, and a point $y \in Y$ such that $h(y) = x$ and $dh_y : T_y Y \to T_x X$ is surjective. (In the proof of (b), it suffices to assume that Y satisfies LSAP at y.) We say that X is *strongly dominable* by LSAP manifolds if the above condition holds at every point $x \in X$.

We now show the following functorial properties of LSAP. Recall also that every LSAP manifold is an Oka-1 manifold; see Proposition 5.4 (a).

Proposition 5.5. *Let X and Y be complex manifolds.*

  (i) *If X is strongly dominable by LSAP manifolds, then X is an LSAP manifold. In particular, if Z is an LSAP manifold and $Z \to X$ is a surjective holomorphic submersion, then X is an LSAP manifold.*
 (ii) *If $h : X \to Y$ is a surjective Oka map and Y is an LSAP manifold, then X is an LSAP manifold.*
(iii) *LSAP is a Hausdorff-local property, that is, if a complex manifold X is covered by sets that are open in the Hausdorff topology and satisfy LSAP, then X satisfies LSAP.*

Proof. The proof of (i) is similar to the proof of part (b) in Proposition 5.4; we include the details since they were not given in [2].

Pick a point $x \in X$. By the hypothesis there are a holomorphic map $h : Y \to X$ and a point $y \in Y$ such that $h(y) = x$, h is submersive at y, and Y satisfies LSAP at y. Let $U_0 \subset Y$ be a neighbourhood of y such that the condition in Definition 5.2 holds for sprays of discs in Y with values in U_0. Since h is a submersion at y, the implicit function theorem gives a smaller neighbourhood $U \subset U_0$ of y such that the restriction $h : U \to V := h(U) \subset X$ is isomorphic to the projection $U \cong V \times W \to V$ by a fibre-preserving map, where W is a neighbourhood of y in the fibre $Z = h^{-1}(x)$. In this local trivialisation we write $y = (x, z)$ with $z \in Z$. Let $D \subset D'$ be a special pair of discs (see Definition 5.1) and $F : D \times \mathbb{B}^N \to V$ be a holomorphic spray. Let $G = (F, z) : D \times \mathbb{B}^N \to U \cong V \times W$ be the lifting of F with the constant second component z. By the assumption, there is a number $0 < r < 1$ such that G can be approximated on $D \times r\mathbb{B}^N$ by holomorphic maps $\widetilde{G} : D' \times r\mathbb{B}^N \to Y$. The holomorphic map $\widetilde{F} = h \circ \widetilde{G} : D' \times r\mathbb{B}^N \to X$ then approximates F on $D \times r\mathbb{B}^N$. This shows that X has LSAP at the point x. Since x was arbitrary, X has LSAP.

To prove (ii), fix a point $x \in X$ and set $y = h(x) \in Y$. Choose a neighbourhood $V \subset Y$ of y such that the condition in Definition 5.2 holds for sprays of discs with values in V. Pick a neighbourhood $U \subset X$ of x such that $h(U) \subset V$. Let $D \subset D' \subset \mathbb{C}$ be a special pair of discs and $F : D \times \mathbb{B}^N \to U$ be a holomorphic spray. By the assumption, there is a number $0 < r < 1$ such that the spray $G = h \circ F : D \times \mathbb{B}^N \to V$ can be approximated on $D \times r\overline{\mathbb{B}}^N$ by holomorphic maps $\widetilde{G} : D' \times r\overline{\mathbb{B}}^N \to Y$. Assuming that the approximation is close enough, we see as in [16, proof of Theorem 3.15] that there is a holomorphic map $\widetilde{F} : D \times r\overline{\mathbb{B}}^N \to X$ which is uniformly close to F on $D \times r\overline{\mathbb{B}}^N$ and satisfies $h \circ \widetilde{F} = \widetilde{G}$, that is, $\widetilde{F}$ is a lifting of $\widetilde{G}$ on $D \times r\overline{\mathbb{B}}^N$ with respect to the map $h : X \to Y$. (The construction of $\widetilde{F}$ uses a holomorphic family of holomorphic retractions on the fibres of h, provided by [16, Lemma 3.16].) Since h is a topological fibration, $\widetilde{F}$ extends to a continuous lifting of $\widetilde{G}$ on $D' \times r\overline{\mathbb{B}}^N$. Finally, since $h : X \to Y$ is an Oka map, we can approximate $\widetilde{F}$ on $D \times r\overline{\mathbb{B}}^N$ by holomorphic liftings

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$F' : D' \times r\overline{\mathbb{B}}^N \to X$ of $\widetilde{G} : D' \times r\overline{\mathbb{B}}^N \to Y$. This shows that X has LSAP at the point x. Since x was arbitrary, we conclude that X is an LSAP manifold.

Assertion (iii) is evident from the definition of LSAP and is also a consequence of (i). $\quad\square$

Since a holomorphic fibre bundle with Oka fibre is an Oka map, we have the following corollary to Proposition 5.5.

Corollary 5.6. *If $X \to Y$ is a holomorphic fibre bundle with Oka fibre, then X has LSAP if and only if Y has LSAP. This holds in particular if $X \to Y$ is a holomorphic covering map.*

Problem 5.7. Does the conclusion of Corollary 5.6 hold under the weaker assumption that the fibre of $X \to Y$ is an LSAP manifold?

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