Integration by parts - examples & solutions

① $\displaystyle\int \frac{3-x}{3e^x}\,dx = \int (3-x)\left(\tfrac{1}{3}e^{-x}\right)dx$

$\displaystyle = \frac{1}{3}\int \underset{u}{(3-x)}\ \underset{v'}{e^{-x}}\,dx$

$u = 3-x \quad v = -e^{-x}$

$u' = -1 \quad v' = e^{-x}$

$\displaystyle = \frac{1}{3}\left[(3-x)(-e^{-x}) - \int e^{-x}\,dx\right]$

$\displaystyle = \frac{-3+x}{3e^x} - \frac{1}{3}\frac{e^{-x}}{-1} + C$

$\displaystyle = \frac{x-3}{3e^x} + \frac{1}{3e^x} + C.$

② $\displaystyle\int \underset{v'}{x^3}\,\underset{u}{\ln(x)}\,dx = \frac{x^4\ln(x)}{4} - \frac{1}{4}\int \frac{x^4}{x}\,dx$

$u = \ln(x) \quad v = \frac{x^4}{4}$

$u' = \frac{1}{x} \quad v' = x^3$

$\displaystyle = \frac{x^4\ln(x)}{4} - \frac{1}{4}\int x^3\,dx$

$\displaystyle = \frac{x^4\ln(x)}{4} - \frac{1}{16}x^4 + C.$

or

D		I
$\ln(x)$	+	x^3
$1/x$	−	$\frac{x^4}{4}$

$\displaystyle\int x^3\ln(x)\,dx = \frac{x^4\ln(x)}{4} - \int\left(\frac{1}{x}\right)\left(\frac{x^4}{4}\right)dx$

$\displaystyle = \frac{x^4\ln(x)}{4} - \frac{1}{16}x^4 + C.$

(3)

$$\int \underset{v'}{e^x} \underset{u}{\sin(3x)}\,dx = e^x \sin(3x) - 3\int \underset{v'}{e^x} \underset{u}{\cos(3x)}\,dx$$

$$\boxed{\begin{array}{ll} u = \sin(3x) & v = e^x \\ u' = 3\cos(3x) & v' = e^x \\ \hline u = \cos(3x) & v = e^x \\ u' = -3\sin(3x) & v' = e^x \end{array}}$$

$$= e^x \sin(3x) - 3\left[e^x \cos(3x) + 3\int e^x \sin(3x)\,dx\right]$$

$$= e^x \sin(3x) - 3e^x \cos(3x) - 9\int e^x \sin(3x)\,dx$$

$$\int e^x \sin(3x)\,dx = e^x \sin(3x) - 3e^x\cos(3x) - 9\int e^x \sin(3x)\,dx$$

so

$$10\int e^x \sin(3x)\,dx = e^x\left(\sin(3x) - 3\cos(3x)\right)$$ [adding $9\int e^x \sin 3x\,dx$ to both sides]

$$\int e^x \sin(3x)\,dx = \frac{e^x\left(\sin(3x) - 3\cos(3x)\right)}{10} + C$$

(4) Find the area between $y=(x+1)\ln(x)$ and the x-axis between $x=1$ & $x=e$.

$$A = \int_1^e \underset{v'}{(x+1)}\,\underset{u}{\ln(x)}\,dx \qquad u=\ln(x) \quad u'=\frac{1}{x} \qquad v=\frac{x^2}{2}+x \quad v'=x+1$$

$$= \left[\left(\frac{x^2}{2}+x\right)\ln(x)\right]_1^e - \int_1^e \frac{\frac{x^2}{2}+x}{x}\,dx$$

$$= \left[\left(\frac{x^2}{2}+x\right)\ln(x)\right]_1^e - \int_1^e \frac{x}{2}+1\,dx$$

$$= \left[\left(\frac{x^2}{2}+x\right)\ln(x)\right]_1^e - \left[\frac{x^2}{4}+x\right]_1^e$$

$$= \left(\frac{e^2}{2}+e\right)(1) - \left(\frac{1}{2}+1\right)(0) - \left[\left(\frac{e^2}{4}+e\right) - \left(\frac{1}{4}+1\right)\right]$$

$$= e\left(\frac{e}{2}+1\right) - e\left(\frac{e}{4}+1\right) + \frac{5}{4}$$

$$= \frac{3e^2}{4} + \frac{5}{4}. \quad \blacksquare$$