Midterm I solutions

(1)

(a) $\int_0^6 f(x)\,dx = 10$, $\int_0^4 f(x)\,dx = 7$

(i) $\int_4^6 f(x)\,dx = \int_0^6 f(x)\,dx - \int_0^4 f(x)\,dx$ [1]

$= 10 - 7$

$= 3$

(ii) $\int_6^0 f(x)\,dx = -\int_0^6 f(x)\,dx$ [1]

$= -10$

(iii) $\int_0^4 2f(x)\,dx = 2\int_0^4 f(x)\,dx$ [1]

$= 14.$

(b) $f''(t) = t^{-2}$

so $f'(t) = \int f''(t)\,dt$

$= \int t^{-2}\,dt$

$= -t^{-1} + A.$

$f(t) = \int f'(t)\,dt$

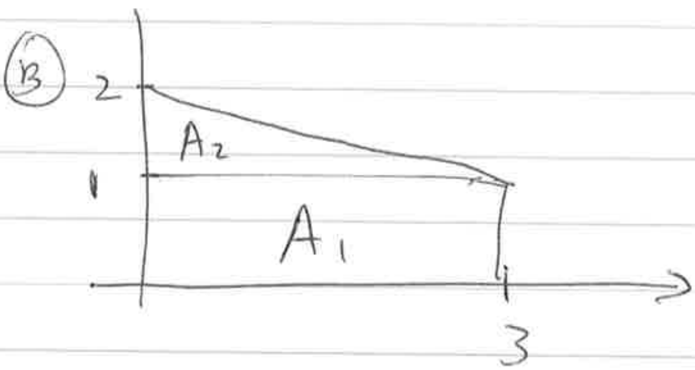
$= \int -t^{-1} + A\,dt$

$$\boxed{f(t) = -\ln|t| + At + C.}$$ [5]

(c) A by symmetry $\int_0^4 f(x)\,dx = 0$

$$\left(\text{as } \int_0^2 f(x)\,dx = -\int_2^4 f(x)\,dx \right).$$ [1]

B



$$\int_0^3 f(x)\,dx = A_1 + A_2$$

$$= 1 \times 3 + \frac{1}{2} 1 \times 3 \qquad \text{(rectangle + triangle)}$$

$$= 3 + \frac{3}{2}.$$

$$= \frac{9}{2}.$$ [2]

(d) $\displaystyle\int \frac{\sin(1/x)}{x^2}\,dx$

$u = 1/x = x^{-1}$

so $\dfrac{du}{dx} = -x^{-2}$

$du = -\dfrac{1}{x^2}\,dx.$

$$= -\int \underbrace{\sin\left(\tfrac{1}{x}\right)}_{\sin(u)} \underbrace{\left(-\frac{1}{x^2}\,dx\right)}_{du.}$$

$$= -\int \sin(u)\,du$$

$$= -(-\cos(u)) + C$$

$$= \cos\left(\frac{1}{x}\right) + c.$$

[4]

(e) $f(x) = 3x e^{3x^2}$

$$A = \int_1^3 3x e^{3x^2}\,dx$$

$$= \frac{1}{2}\int_1^3 6x e^{3x^2}\,dx$$

let $u = 3x^2$

$du = 6x\,dx.$

$$= \frac{1}{2}\int_1^3 \underbrace{e^{3x^2}}_{e^u} \underbrace{(6x\,dx)}_{du.}$$

$x = 3$ $u = 27$

$x = 1$ $u = 3.$

$$= \frac{1}{2}\int_3^{27} e^u\,du.$$

$$= \frac{1}{2}\left[e^{27} - e^3\right].$$

[5]

f) $\int \underbrace{3x^2}_{u} \underbrace{e^{3x}}_{v'} dx$ $\qquad u = 3x^2 \quad v = \frac{1}{3}e^{3x}$

$\qquad u' = 6x. \quad v' = e^{3x}$

$= uv - \int vu' \, dx$ (integration by parts)

$= x^2 e^{3x} - \frac{6}{3}\int \underbrace{x}_{u}\underbrace{e^{3x}}_{v'} dx$ $\qquad u = x \quad v = \frac{1}{3}e^{3x}$

$\qquad u' = 1 \quad v' = e^{3x}$

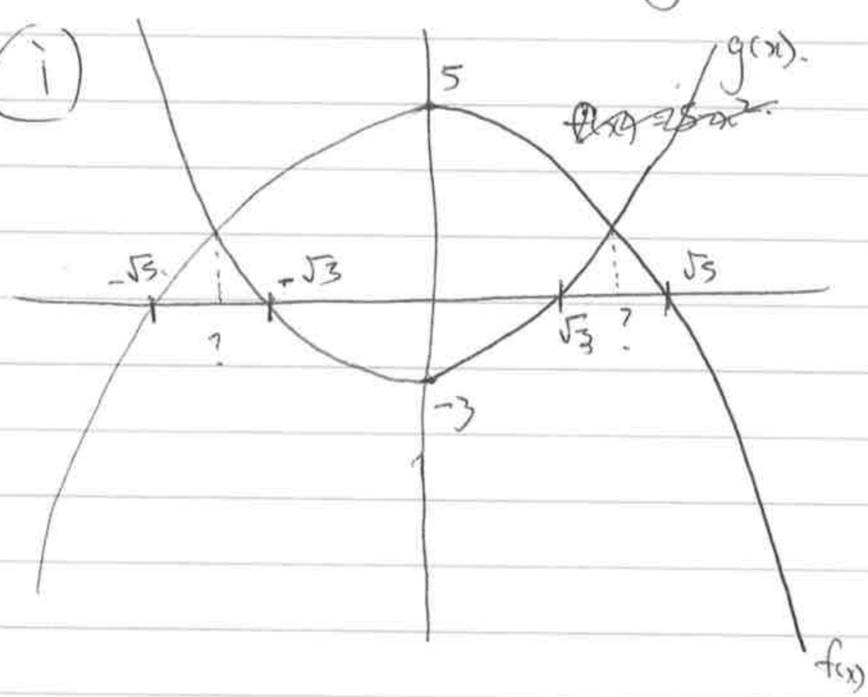
$= x^2 e^{3x} - 2\left[\frac{xe^{3x}}{3} - \frac{1}{3}\int e^{3x} dx\right]$

$= x^2 e^{3x} - \frac{2xe^{3x}}{3} + \frac{2}{9}e^{3x} + c.$ [5]

2 Applications of integration.

(a) $f(x) = 5 - x^2$, $g(x) = x^2 - 3$.

(i)



[2]

(ii) we need to solve

$$5 - x^2 = x^2 - 3.$$
$$8 = 2x^2$$
$$x^2 = 4$$
$$x = \pm 2.$$

[1]

(iii) $$A = \int_{-2}^{2} f(x) - g(x)\, dx$$

$$= \int_{-2}^{2} (5 - x^2) - (x^2 - 3)\, dx$$

$$= \int_{-2}^{2} 5 - x^2 - x^2 + 3\, dx$$

$$= \int_{-2}^{2} 8 - 2x^2\, dx$$

$$= \left[8x - \frac{2x^3}{3} \right]_{-2}^{2}$$

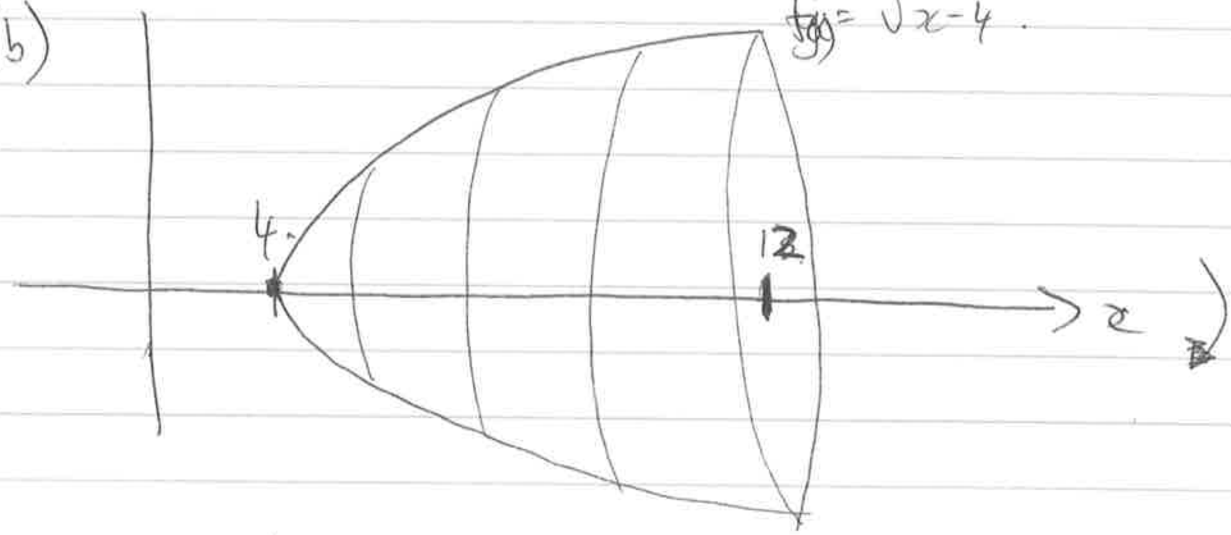
$$= \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right).$$

$$= 32 - \frac{32}{3}$$

$$= 32\left(1 - \frac{1}{3}\right) = 32\left(\frac{2}{3}\right) = \frac{64}{3}.$$

[3]

(b)



$$V = \pi \int_a^b \left(f(x)\right)^2 dx$$

$$= \pi \int_4^{12} \left(\sqrt{x-4}\right)^2 dx$$

$$= \pi \int_4^{12} x - 4 \, dx$$

$$= \pi \left[\frac{x^2}{2} - 4x \right]_4^{12}$$

$$= \pi \left(\frac{144}{2} - 48 \right) - \pi \left(8 - 16 \right)$$

$$= \pi (72 - 48) + 8\pi$$

$$= 24\pi + 8\pi$$

$$= 32\pi$$

[5]

(c) $T(t) = 53 - 19\cos(0.017t - 1)$.

(i) $$\overline{T} = \frac{1}{60}\int_0^{60} T(t)\,dt$$

$$= \frac{1}{60}\int_0^{60} 53 - 19\cos(0.017t-1)\,dt$$

$$= \frac{1}{60}\left[53t - \frac{19}{0.017}\sin(0.017t-1)\right]_0^{60}$$

$$= \frac{1}{60}\left[53(60) - \frac{19}{0.017}\sin(60\times 0.017-1)\right]$$

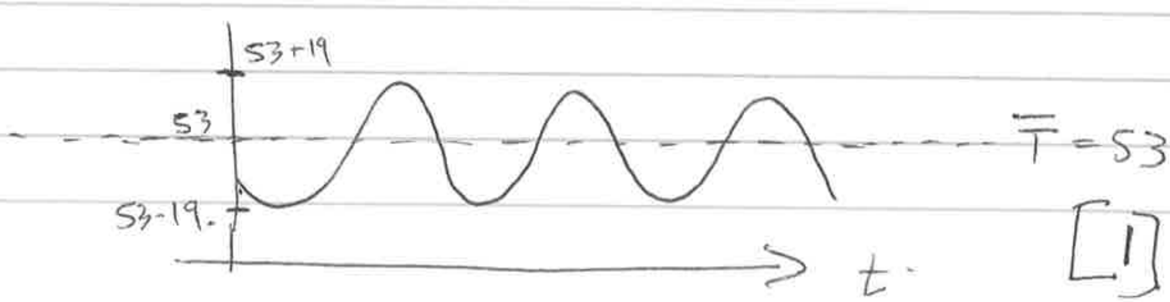
$$- \frac{1}{60}\left[0 - \frac{19}{0.017}\sin(-1)\right]$$

$=$

[3]

(ii) You could calculate the same integral here, and end up with $\overline{T} \approx 53$, or notice that the average value of the cosine function is 0, so the average should just be

$$\overline{T} = 53.$$



[1]