At least four doors, numerous goats, a car, a frog, four lily pads and some probability

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Undergraduate Seminar

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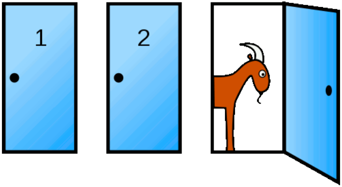
Wednesday 13 October 2010

Consider the following two problems...

Let's make a deal



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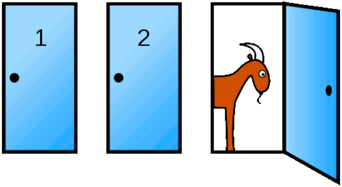

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After you choose one of the doors, I open one of the other two doors to reveal a goat.

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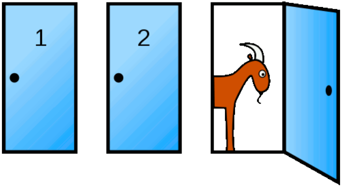

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After you choose one of the doors, I open one of the other two doors to reveal a goat.

Would you like to stick with your original choice of door, or change?

Consultation hour

The number of students queued outside my office at the start of my consultation hour, X, is well approximated by a Poisson distribution with parameter 2:

$$\Pr(X = j) = e^{-2}\frac{2^j}{j!}, \quad j \in \{0, 1, 2, \ldots\}.$$

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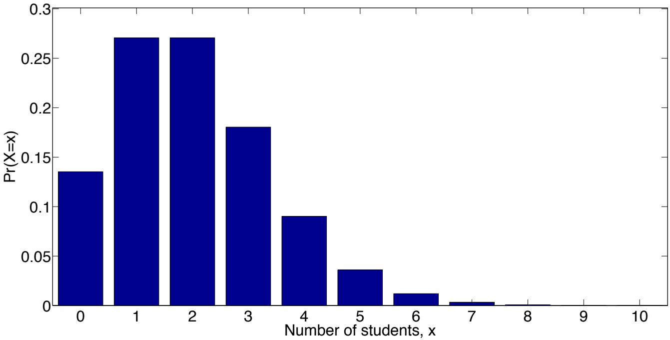

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What is the probability I open my door to find 2 students waiting, following a knock on my door, at the start of my consultation hour?

Conditional probability

Let A and B be two events. Then

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

Consultation hour

Given I hear a knock on the door, I know that there must be at least 1 student waiting:

$$\begin{aligned}\Pr(X = j | X > 0) &= \frac{\Pr(X = j \cap X > 0)}{\Pr(X > 0)} \\ &= \frac{\Pr(X = j)}{1 - \Pr(X = 0)}, \;\; j \in \{1, 2, \ldots\}.\end{aligned}$$

So the probability I open my door to find 2 students waiting is

$$\frac{\Pr(X = 2)}{1 - \Pr(X = 0)} = \frac{2e^{-2}}{1 - e^{-2}} \approx 0.3130$$

(compare with $\Pr(X = 2) = 2e^{-2} \approx 0.2707$).

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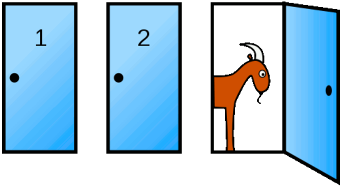

Consider the possible allocations of goats/car to doors:

Allocation	Door 1	Door 2	Door 3
1	Car	Goat	Goat
2	Goat	Car	Goat
3	Goat	Goat	Car

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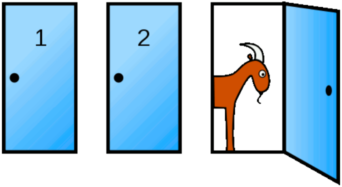

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Say you pick door 1 (any door will do!) – and let's consider each scenario.

Let's make a deal

Let A and B be two events. Then

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

Using this, we could evaluate

$$\Pr(H = h | C = c, S = s)$$

where
C be the RV corresponding to the door with the car,
S be the RV corresponding to the door you select, and
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But we want to know

$$\Pr(C = c | H = h, S = s).$$

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$$\begin{aligned}\Pr(B|A) &= \frac{\Pr(A \cap B)}{\Pr(A)} \\ &= \frac{\Pr(A \cap B)}{\Pr(A \cap B) + \Pr(A \cap B')} \\ &= \frac{\Pr(A|B)\Pr(B)}{\Pr(A|B)\Pr(B) + \Pr(A|B')\Pr(B')}\end{aligned}$$

Generally, $B_j,\ j = 1, \ldots, n$,

$$\Pr(B_j|A) = \frac{\Pr(A|B_j)\Pr(B_j)}{\sum_{i=1}^{n} \Pr(A|B_i)\Pr(B_i)}$$

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$$\Pr(C=c|H=h,S=s)=\frac{\Pr(H=h|S=s,C=c)\Pr(C=c)}{\sum_{j=1}^{3}\Pr(H=h|S=s,C=j)\Pr(C=j)}.$$

So, for example,

$$\begin{aligned}\Pr(C=1|H=2,S=1) &= \frac{\Pr(H=2|S=1,C=1)}{\sum_{j=1}^{3}\Pr(H=2|S=1,C=j)}\\ &= \frac{\frac{1}{2}}{\frac{3}{2}}=\frac{1}{3}\end{aligned}$$

and

$$\Pr(C=3|H=2,S=1)=\frac{2}{3}.$$

Discrete time Markov chains

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We can form a matrix P, with entry (i, j) telling us the probability of jumping from pad i to pad j: for example,

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix}.$$

Discrete time Markov chains

So given

$$p(n) = (\Pr(X_n = 1), \Pr(X_n = 2), \Pr(X_n = 3), \Pr(X_n = 4)),$$

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This distribution, π, is equal to the *limiting distribution* (for certain DTMCs as considered here):

$$\lim_{n\to\infty} p(n) = p(0) \lim_{n\to\infty} P^n = \pi.$$

Discrete time Markov chains

For our friendly frog, we have

$$\pi = (0.2, 0.3, 0.3, 0.2)^T.$$

More goats... well, less goats and then more goats

Now we wish to model the number of female goats (henceforth called goats) in a paddock at the end of each breeding cycle, X_t, where $X_t \in \{0, 1, \ldots, N\}$ for $t = \{0, 1, \ldots\}$, and N is the known maximum number of goat that can fit in the paddock.

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$$\Pr(X_{(t+1)^-} = n) = \binom{m}{m-n} d^{(m-n)}(1-d)^n \quad (n \leq m).$$

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We can place these probabilities in a matrix D (as we did for P for the frog, before).

More goats

Now, if there are $X_{(t+1)^-} = n$ goats just before birth, there is $X_{(t+1)} = k$ goats following birth with probability

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for $k = n, \ldots, \min\{2n, N-1\}$, and if $n \geq N/2$,

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Then, the matrix P for our model of goat dynamics, is:

$$P = DB,$$

the matrix product of our matrix D and our matrix B.

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If we evaluate the stationary (limiting) distribution of our goat model, we find:

$$\pi = (1, 0, 0, \ldots, 0)$$

where π is of length N.

In the long-term we will encounter a time at which we have no goats. Obviously, with no goats, we will remain with no goats forever.

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But what happens if we look at *realisations* of this model...

Some goats, for a very long time...

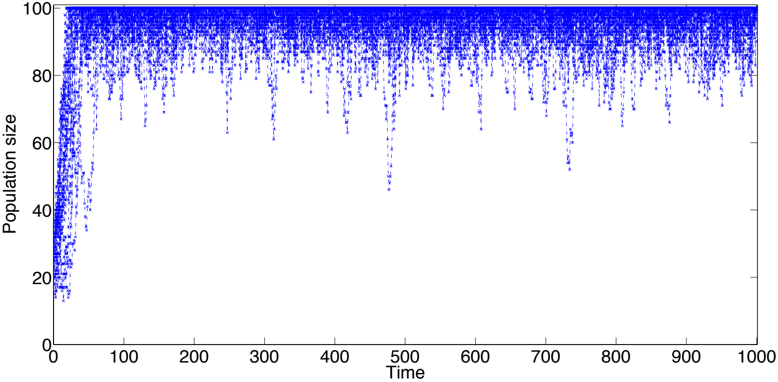

Figure: 20 simulations with $N = 100$, $b = 0.4$ and $d = 0.25$.

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This is a *conditional* distribution—conditioned on non-extinction.

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This distribution can be shown to equal:

$$\pi^* P^* = \rho \pi^*$$

where P^* is the matrix P with row and column corresponding to the *absorbing state* 0 removed, and ρ is the real, maximum-modulus eigenvalue of P^* and π^* is the left eigenvector (normalised to sum to 1) corresponding to ρ.

Quasi-stationarity

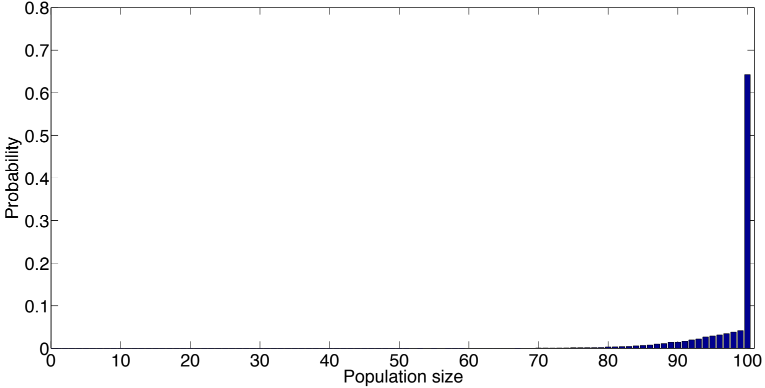

Figure: Approximate QSD based upon simulations from $t = 500 \to 1000$.

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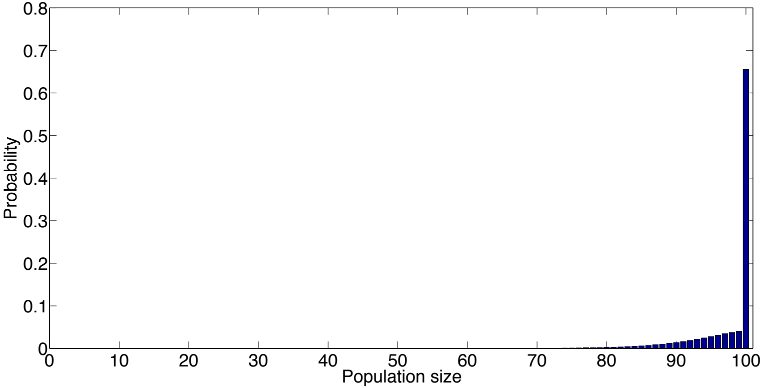

Figure: Exact QSD.

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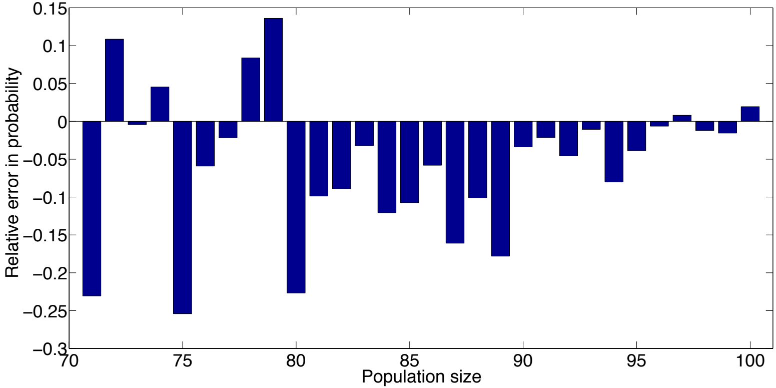

Figure: Relative error in QSD.

Current research

1. Using these types of methods to determine optimal strategies for prolonging species persistence.
2. Developing methods for approximating these distributions, to enable (efficient) evaluation.
3. Discovering results which allow us to determine if these distributions *exist* and if they are *unique*, and what they correspond to if not limiting or unique.
4. . . . and much much more . . .

Take home messages

1. Importance of conditioning on information.
2. Usefulness of probability, and in particular Markov chains, for (biological) modelling.
3. Many open research questions in this field.

Thanks!

Thanks for your attention!

http://www.maths.adelaide.edu.au/joshua.ross