For the love of logs

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For the love of dogs?

Outline

Tycho Brahe

- Ptolemaic vs Copernican models
- The observatory at Hven
- Brahe's revolutions in observation

Kepler

- The battle with Mars (and only Mars)
- Normalization
- Circles and lines
- Law 1: Planets move in ellipses, with the sun at one focus
- Law 2: Planets sweep out equal areas in equal times

The Lean Years: Prosthaphaeresis

- No breakthroughs for 12 years
- To multiply: $\cos A \cos B = 1/2(\cos(A+B) + \cos(A-B))$
- Look up cos tables and arccos tables
- Mars alone consumed 1000 pages

Napier

Napier, along with many others, noticed that when multiplying powers of integers you simply added the powers: $2^a 2^b = 2^{a+b}$. But the gaps between, say, $2^6 = 64$ and $2^7 = 128$ were discouraging. He noted that a small base would “close up” the gaps, and so defined $N_L(n)$, the Napier log of n by

$$n = 10^7 \left(1 - \frac{1}{10^7}\right)^{N_L(n)}.$$

- Note that $N_L(10^7) = 0$ and $N_L(9999999) = 1$.
- $N_L(mn/10^7) = N_L(m) + N_L(n)$.

Briggs

- Henry Briggs met with Napier to discuss the awkwardness
- Redefine so that $\log_{10}(1) = 0$ and $\log_{10}(10) = 1$. That is

$$n = 10^{\log_{10}(n)}$$

- Logarithms were immediately successful
- Mechanical devices proliferated: slide rules by 1622
- Survived until 1970's (computers and pocket calculators)

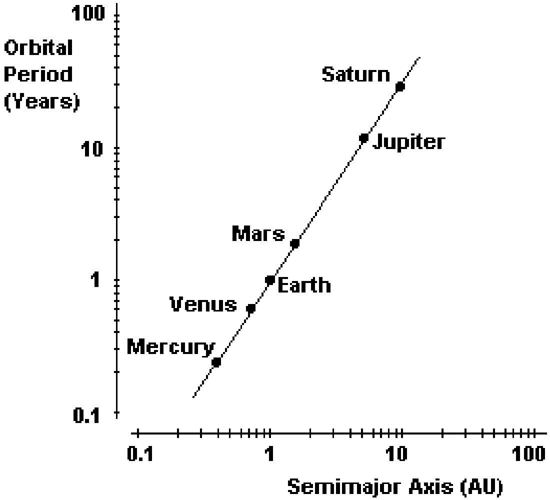
Music of the ellipsoids

- Kepler delighted in the simplification of logarithms
- Examining the data for periods of planets and their distance from the sun he "at first thought I was dreaming"

  > *...the proportion between the periodic times of any two planets is precisely the sesquialterate proportion of their mean distances*

- That is, $T \propto R^{3/2}$.

Sesquialterity



Power Laws

Recall the basic operation of logs: if y and x satisfy

$$y^{k_1} = Ax^{k_2} \text{ then } \quad \log y = \frac{k_2}{k_1} \log x + \frac{\log A}{k_1},$$

a linear relationship. The availability of logs (and the calculational facility they allow) seem here to have produced an important physical law that was previously hidden amongst the data.

Music among the primes

- Patterns and/or predictability among the prime numbers?
- A "Gaussian blur" showed that if $\pi(n)$ is the number of primes less than n then

$$\lim_{n\to\infty} \frac{\pi(n)}{\frac{n}{\ln n}} = 1.$$

- That is, $\pi(n) \sim \frac{n}{\ln n}$, so near n one out of every $\ln n$ numbers (on average) is prime.
- We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but excruciatingly slowly. $\sum_{n=1}^{k} \frac{1}{n} \sim \int_1^k 1/x = \ln k$. What about

$$\sum_{\text{primes} \le k} \frac{1}{n}?$$

Slower and slower...

- Approximate, for $n > 3$ say, by

$$\sum_{n=3}^{k} \frac{1}{n} \frac{1}{\ln n}$$

as the "expected contribution" to the sum is $\frac{1}{n \ln n}$.

- Approximate this by

$$\int_3^k \frac{1}{x \ln x} dx$$

- Which equals... $[\ln(\ln x)]_3^k \sim \ln(\ln k)$
- So the sum diverges, but oh so slowly. For $k = 10^{100}$ (a googol), it is about $\ln(100 \ln(10)) = \ln(430) \sim 6$. Even for a googolplex terms (10 to a googol) it has barely raised a whimper... it's about 430 at that stage.

Stirling's Formula

Calculating $n!$ for large n is **nasty**. For a very modest $n = 200$ we have...

7886578673647905035523632139321850622951359776871732632947425332443594499634033429203042840119846239041772121389196388302576427902426371050619266249528299311134628572707633172373969889439224456214516642402540332918641312274282948532775242424075739032403212574055795686602260319041703240623517008587961789222227896237038973747200000000000000000000000000000000000000000000000000

For $n = 10000$ we're already at 35660 digits.

Logs?

Obviously we can look at $\ln(n!) = \ln n + \ln(n-1) + \cdots + \ln(1)$, and approximate the sum by an integral again

$$\sum_{k=1}^{n} \ln(k) \sim \int_1^n \ln(x)dx = [x\ln(x) - x]_1^n = n\ln n - n + 1.$$

but lets take another approach. Instead note that

$$\begin{aligned} n! &= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)^2\left(\frac{3}{4}\right)^3\cdots\left(\frac{n-1}{n}\right)^{n-1} n^n \\ &= n^n \prod_{j=1}^{n-1}\left(\frac{j}{j+1}\right)^j \\ &= \frac{n^n}{\prod_{j=1}^{n-1}\left(1+\frac{1}{j}\right)^j}. \end{aligned}$$

e

Each term $(1 + 1/j)^j$ tends to e as $j \to \infty$, so $n! \sim \frac{n^n}{e^{n-1}}$ as per the previous integration. But this convergence to e is quite slow, and the combination $(1 + 1/j)^{j+1/2}$ is much swifter. Note that

$$\begin{aligned}(1+1/j)^{j+1/2}/e &= \exp((j+1/2)(\ln(1+1/j)-1)) \\ &= \exp((j+1/2)(1/j - 1/2j^2 + 1/3j^3 + \cdots) - 1) \\ &\sim \exp(1/12j^2)\end{aligned}$$

We've used Mercator's series for $\ln(1 + x)$:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Toward Stirling

Iterating the previous expression for $n!$ gives

$$\begin{aligned} n! &= \left(\frac{1}{2}\right)^{1.5}\left(\frac{2}{3}\right)^{2.5}\left(\frac{3}{4}\right)^{3.5}\cdots\left(\frac{n-1}{n}\right)^{n-.5} n^{n+1/2} \\ &= \frac{n^{n+1/2}}{\prod_{j=1}^{n-1}\left(1+\frac{1}{j}\right)^{j+1/2}} \\ &\sim \frac{n^{n+1/2}}{e^{n-1}} \end{aligned}$$

This differs from Stirling's approximation

$$n! \sim \frac{\sqrt{2\pi}\, n^{n+1/2}}{e^n}$$

by a simple factor of $\sqrt{2\pi}/e \sim 1.08$. This 8% is hard work!

Suggestion

Play with the previous expression for $n!$ and try to show that you can improve Stirling's formula enormously by using:

$$n! \sim \frac{\sqrt{2\pi}\, n^{n+1/2}}{e^n} e^{\frac{1}{12n}}.$$

If nothing else, try this formula for various (even very small) n to see its spectacular agreement with $n!$.

The utterly straightforward

Suppose we wish to calculate $\log_{10} n$. First note that the magic of the log function gives

$$\log_{10} n = \log_{10} p_1 + \log_{10} p_2 + \cdots + \log_{10} p_k \text{ for } n = p_1 p_2 \ldots p_k,$$

so we only need consider prime numbers. For example

$$2^{10} = 1024 \sim 1000 = 10^3$$
$$3^4 = 81 \sim 80 = 2^3 \times 10$$

- The first gives $\log 2 = 3/10 = .3$, which is close to the exact .30103.
- The second gives $\log 3 = \frac{3 \log 2 + 1}{4} = .475$, compared with the exact value of .4771. Good, but not great.

Solving gives...

We can do much better. First look for better coincidences, involving say 2,3,7 and 11. (Why no 5?)

$$21^2 = 3^2 \times 7^2 = 441 \sim 440 = 20 \times 22 = 10 \times 2^2 \times 11,$$

so we have $2\log 3 + 2\log 7 = 1 + 2\log 2 + \log 11$.
Similar expressions (eg $3^4 \times 11^2 = 9801 \sim 9800 = 2 \times 7^2 \times 10^2$) using the difference of two squares give 4 linear equations for the four unknowns.

$$\begin{aligned}
\log 2 &= .30125(\text{ actual is } .30102) \\
\log 3 &= .47698(\text{ actual is } .47712) \\
\log 7 &= .84518(\text{ actual is } .84509) \\
\log 11 &= 1.04184(\text{ actual is } 1.04139)
\end{aligned}$$

Corrections using series

These are good, but simple corrections help!

$$\ln(1+x) \sim x, \text{ so } \log(1+x) \sim .4343x$$

(.4343 is just $\log_{10} e$). So instead we use expressions like

$$\log\left(\frac{1024}{1000}\right) \sim .4343 \times \frac{24}{1000}$$

which gives $\log 2 = .301042($ actual $is.3010299)$

Exercise: Play with the simultaneous equations to get values for $\log 2, \log 3, \log 7$ and $\log 11$.

One Final Coincidence

Taking $.7 = 7/10$ and squaring it, then squaring the result, and continuing to do this 9 times gives

$$\frac{7^{2^9}}{10^{2^9}} \sim 4.90046 \times 10^{-80}.$$

So we get $\frac{7^{512}}{10^{512}} \sim 7^2 \times 10^{-81}$, which yields

$$7^{510} \sim 10^{431}.$$

Calculating gives

$$\log 7 \sim \frac{431}{510} = .8450980392(\text{ actual is } .8450980400)$$

Reference

The approach to calculating logs in the “Desert Island Logs” section is drawn from N. David Mermin’s highly recommended “Boojums all the way through”.