Spot the Difference:
How to tell when two things are the same
(and when they’re not!)

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Outline

Introduction

Topology

Category Theory
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Category Theory
## Examples of Mathematical Structures

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<td>(( V, +, \mathbb{R}\text{-mult}))</td>
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<td>Functions on ( \mathbb{R} )</td>
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<td>(non)-example: (( \mathbb{R}^3, \times ) \xrightarrow{\sim} \text{Lie algebra})</td>
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<td>( \mathbb{R} ) (cts properties)</td>
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Sameness Problems

Problem
How to come up with a natural notion of “sameness” for these (& other) structures?

▶ “=” is usually too strict

e.g. \( \{ x-y \text{ plane} \} \subseteq \mathbb{R}^3 \)

\[ \mathbb{R}^2 \]

\[ \{ x-y \} \neq \mathbb{R}^2 \quad \text{but} \quad \{ x-y \} \simeq \mathbb{R}^2 \]

Similarly: any 2D subspace of any \( \mathbb{R}^n \simeq \mathbb{R}^2 \)
Sameness Problems

Key Idea

*Change object but preserve structure*

Want maps $V \rightarrow W$ which preserve the vector space (or group, or algebra, or . . .) structure

- e.g. **NOT** $\mathbb{R}^3 \rightarrow \mathbb{R}^2$; $(x, y, z) \mapsto (x^2 y, \sin(z)/\sqrt{x})$

Examples:

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Equivalence: Find maps both ways (i.e. maps with inverses)

Problem

*How can you tell if 2 objects are equivalent or not?*

- $A \simeq B$: Find one
- $A \not\simeq B$: Extract info which doesn’t change under equivalence and show it’s different for $A$ and $B$

$\leadsto$ invariants

Example: (finite-dimensional) vector spaces:

\[ V \simeq W \iff \dim V = \dim W \]
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**Topology**

**Idea:** Throw away notions of measurement, distances, etc. . .
Keep ideas of “closeness” and “connectedness”

**Example:**
Equivalency condition: Continuous deformations

\( (\text{homeomorphisms}) \)

\[ X \xrightarrow{f} Y \quad \text{and} \quad Y \xrightarrow{f^{-1}} X \text{ continuous} \]

Example: \( \mathbb{R} \to \mathbb{R}; \ x \mapsto x^3 \) vs \( \mathbb{R} \to \mathbb{R}; \ x \mapsto \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases} \)
Manifolds

Manifolds are important examples of topological spaces

Roughly: A manifold is a topological space which locally looks like (i.e. is homeomorphic to) $\mathbb{R}^n$

Examples:

1D $S^1$

2D $S^2$

$\mathbb{T}^2$

Other surfaces

3D $S^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}$

4D The Universe!

10D The Universe?
Question
How can you tell if 2 manifolds are homeomorphic?

Answer: Look for invariants!

Example Question: $S^2 \not\cong \mathbb{T}^2$

Note: It’s not necessarily obvious. Consider
The Fundamental Group

Consider loops:
\[ \{ \gamma : [0, 1] \to X \mid \gamma(0) = \gamma(1) = x_0 \} \]

\[ \gamma_1 \sim \gamma_2 \iff \gamma_1 \xrightarrow{\text{cts deform}} \gamma_2 \]

e.g. \( \gamma \not\sim \gamma' \)
The Fundamental Group

Define:

- $\gamma_1 \ast \gamma_2 = \gamma_1$ then $\gamma_2$

- $\gamma^{-1} = \gamma$ backwards

$\implies$ It’s a group!

The Fundamental Group, $\pi_1(X)$

Important Property: $X \simeq Y \implies \pi_1(X) \simeq \pi_1(Y)$
The Fundamental Group

Back to question: \( S^2 \cong \mathbb{T}^2 \) \( \rightsquigarrow \) is \( \pi_1(S^2) \cong \pi_1(\mathbb{T}^2) \)

\( \pi_1(S^2) : \)

\[ \xrightarrow{\quad} \]

\[ \xrightarrow{\quad} \]

\[ \xrightarrow{\quad} \]

\[ \xrightarrow{\quad} \]

\[ \Rightarrow \pi_1(S^2) \cong \{0\} \quad \text{(i.e. } S^2 \text{ is simply connected)} \]

\( \pi_1(\mathbb{T}^2) : \)

\[ \& \]

\[ \Rightarrow \pi_1(\mathbb{T}^2) \cong \mathbb{Z} \oplus \mathbb{Z} \]

\[ \pi_1(S^2) \ncong \pi_1(\mathbb{T}^2) \Rightarrow S^2 \ncong \mathbb{T}^2 \]
Classification of Surfaces

In general $\pi_1$ doesn’t completely characterise $X$

- 2D: it does

Interesting Question: What about higher dimensions?

e.g. $\pi_1(X^3) = 0 \Rightarrow X^3 \simeq S^3$
More about $\pi_1$

Important Property: $f: X \to Y$ induces $f_* : \pi_1(X) \to \pi_1(Y)$

$\pi_1$ is an example of a functor
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Category Theory

- Formalism for talking about structures and maps which preserve them
- A category consists of
  - Objects
  - Arrows between them

Examples:
- Top: Topological spaces and continuous maps
- Grp: Groups and group homomorphisms
- Vect: Vector spaces and linear maps
- Cat: Category of Categories!

- Invariants for topological spaces $\longleftrightarrow$ Functors $\text{Top} \to \text{Grp}$

More functors: $\pi_n, H_n$