The Real Thing

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Wednesday, 3 August, 2011

Outline

Counting with the Natural Numbers

- A set S is countable if there is a function f from $\mathbb{N}$ onto S. That is, we can label each $s \in S$ by a different natural number.
- Examples: finite sets, $\mathbb{Z}$
- Positive Rational Numbers:

$$\frac{p}{q} \Longleftrightarrow 3^p 5^q \in \mathbb{N} \text{ for } p, q \in \mathbb{N}.$$

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Finite subsets of $\mathbb{N}$

- Map subsets of $\mathbb{N}$ to binary sequences
- eg $\{1, 3, 4, 9\} \Longleftrightarrow 1011000010000000\ldots$
- Finite subsets of $\mathbb{N}$ (except $\emptyset$) always finish with the "tail" $100000000\ldots$
- The collection of all such subsets is countable. If n is the number of digits before the tail of a particular sequence, there are exactly 2^n such sequences.
- So we can always count to our particular sequence in at most $1 + 2^0 + 2^1 + 2^2 + \cdots + 2^n$ steps.

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General subsets of $\mathbb{N}$

Assume they are countable, so we have a map f from $\mathbb{N}$ *onto* the set of all such sequences.

$$\begin{array}{ccl}
f(1) & = & 0\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ \ldots \\
f(2) & = & 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ \ldots \\
f(3) & = & 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ \ldots \\
f(4) & = & 1\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ \ldots \\
f(5) & = & 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ \ldots \\
\ldots & = & \ldots
\end{array}$$

We can always make a sequence not in this list by flipping the diagonal elements: so it begins

$$1\ 0\ 0\ 0\ 1\ \ldots$$

Uncountable Sets

So we **cannot** have listed all the sequences.

The set of all subsets of $\mathbb{N}$ is uncountable.

There's a map from the set of all such sequences onto $[0,1]$. Given a binary sequence $s = a_1 a_2 a_3 a_4 a_5 \ldots$ define

$$x_s = \sum_{n=1}^{\infty} \frac{a_n}{2^n}.$$

At most two sequences correspond to a single real number. Note, for example, that

$$01111111... = 10000000000\ldots \Longleftrightarrow 1/2.$$

So (small exercise!) the real numbers in $[0,1]$ are also uncountable.

Definition

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A subset S of $\mathbb{R}$ is said to have measure zero if for every $\epsilon > 0$ there are open intervals $\{I_n : n \in \mathbb{N}\}$ such that

$$S \subset \bigcup_{n \in \mathbb{N}} I_n \quad \text{and} \quad \sum_{n=1}^{\infty} \text{length}(I_n) < \epsilon.$$

Examples: a single real number, any finite set.

Countable Sets

If $\{a_n : n \in \mathbb{N}\}$ is countable then take

$$I_1 = (a_1 - \frac{\epsilon}{2^3}, a_1 + \frac{\epsilon}{2^3}), I_2 = (a_2 - \frac{\epsilon}{2^4}, a_2 + \frac{\epsilon}{2^4}), \ldots$$

$$\ldots, I_n = (a_n - \frac{\epsilon}{2^{n+2}}, a_n + \frac{\epsilon}{2^{n+2}}), \ldots$$

$$\begin{aligned}\sum_{n=1}^{\infty} \text{length}(I_n) &= \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} = \frac{\epsilon}{2^2} \sum_{n=0}^{\infty} \frac{1}{2^n} \\ &= \frac{\epsilon}{2^2} \frac{1}{1 - \frac{1}{2}} = \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

Consequences

- So every countable set has measure zero.
- But not *just* countable sets. . .

A property is said to hold for almost every real number if the set of elements for which it does not hold has measure zero.

Probabilistically: a property is true almost certainly if the set of elements for which it fails to hold has measure zero.

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Generating a Random Real

- Assume we have a perfectly fair coin.
- Toss it once, twice, ...
- Faster!
- Note the result is a binary sequence, and hence a real number, say x, in $[0, 1]$.

$$x = 1101000 1 \ldots$$

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Monkey Business, Part 1

SuperCoder


SUPERCODER 2000
Air cooled coding keyboard for professional use.
Done
0
1

Simply Normal (base 2)

Definition

A real number written in binary is simply normal (base 2) if

$$\lim_{n\to\infty} \frac{\#1's \text{ in the first n digits}}{n} = \frac{1}{2}.$$

Examples

.0101010101010101... = 1/3
.1100110011001100 11... = 4/5.

Normal (base 2)

Definition

A real number written in binary is normal (base 2) if every binary string s of length k occurs with limiting frequency 2^{-k}, for every $k \in \mathbb{N}$. That is:

$$\lim_{n\to\infty} \frac{\#\text{occurrences of } s \text{ in the first n digits}}{n} = \frac{1}{2^k}.$$

So "111" occurs 1/8th of the time, "010101" occurs 1/64th of the time (in the limit), and so on.

Testing Normality

- Testing is hopeless!
- How many non-normal numbers are there?

Normal *is Normal*

Theorem

(Borel 1909) Almost every real number is normal.

That is, the set of numbers that are not normal has measure zero. So x, our randomly generated real number is almost certainly *normal*.

The world's best key drive?

What does the real number contain?

- Files
- Movies
- A very helpful Index
- Backups!

The world's best key drive?

Wenger 16999 Giant Swiss Army Knife

The world's worst key drive

So many near misses! Trust no one!

The world's worst key drive

Champernowne's Number

Normal numbers base 10. Every string of length k occurs with limiting frequency 10^{-k}, for all $k \in \mathbb{N}$.

.12345678910111213141516171 8...

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Ohne Titel

Thank you