Bundle gerbes

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Outline

Informal definition of p-gerbe

Very informally a p-gerbe is a geometric object representing $p+2$ dimensional cohomology. The motivating example is $p=0$ and two dimensional cohomology. The geometric objects are then $U(1)$ principal bundles or hermitian line bundles.

A little less informally a p-gerbe $\mathcal{P}$ is an object of some category C. There is a functor $C \to \mathrm{Man}$. If M is the image of $\mathcal{P}$ under this functor we say $\mathcal{P}$ is (a p-gerbe) over M and write $\mathcal{P} \Rightarrow M$.

This functor has to admit pullbacks. If $f: N \to M$ is smooth and $\mathcal{P} \Rightarrow M$ there is a unique p-gerbe $f^*(\mathcal{P}) \Rightarrow N$ and a commuting diagram

$$\begin{array}{ccc} f^*(\mathcal{P}) & \to & \mathcal{P} \\ \Downarrow & & \Downarrow \\ N & \xrightarrow{f} & M \end{array}$$

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Associated to every p-gerbe $\mathcal{P} \Rightarrow M$ there is a characteristic class $c(\mathcal{P}) \in H^{p+2}(M, \mathbb{Z})$ which is natural with respect to pullbacks. That is if $f : N \to M$ and $\mathcal{P} \Rightarrow M$ then

$$c(f^*(\mathcal{P})) = f^*(c(\mathcal{P})) \in H^p(N, \mathbb{Z}).$$

With respect to products and duals we require $c(\mathcal{P}^*) = -c(\mathcal{P})$ and $c(\mathcal{P} \otimes \mathcal{Q}) = c(\mathcal{P}) + c(\mathcal{Q})$.

As elements of a category we know what it means for p-gerbes $\mathcal{P}$ and $\mathcal{Q}$ to be isomorphic and in such a case we require that $c(\mathcal{P}) = c(\mathcal{Q})$. The converse is often not true and there is usually some other notion of equivalence which corresponds precisely to $c(\mathcal{P}) = c(\mathcal{Q})$. We say $\mathcal{P}$ is trivial if $c(\mathcal{P}) = 0$.

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Finally we would like some notion of a connection. A connective structure on a p-gerbe $\mathcal{P}$ is additional structure which allows the definition of the $p+2$ curvature which is a closed, differential $p+2$ form ω on M satisfying

$$r(c(\mathcal{P})) = \left[\frac{1}{2\pi i}\omega\right] \in H^{p+2}(M, \mathbb{R})$$

where $r\colon H^*(M, \mathbb{Z}) \to H^*(M, \mathbb{R})$. A connective structure should also allow us to define holonomy which associates to any closed $p+1$ dimensional, oriented, submanifold $X \subset M$ an element $\mathrm{hol}(X) \in U(1)$. If $X = \partial Y$ (Y oriented) then

$$\mathrm{hol}(X) = \exp\left(\int_Y \omega\right).$$

Note that we could get all the above by associating to $\mathcal{P}$ an appropriate Deligne class or differential character.

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Example

$p = 0$

The motivating example for this theory is $U(1)$ bundles or hermitian line bundles.

- Pullback is pullback.
- Product and dual are tensor product and dual.
- The characteristic class is the first Chern class.
- Equivalence is isomorphism and triviality is triviality.
- The connective structure is a connection.
- The curvature and holonomy are curvature and holonomy.

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Example

$p = -1$

Consider functions $\phi \colon M \to U(1)$. We say $\phi \Rightarrow M$.

- The pullback is pullback of functions $f^*(\phi) = \phi \circ f$.
- The dual is $\phi^*(m) = \phi(m)^{-1}$.
- The product of ϕ and χ is $(\phi \otimes \chi)(m) = \phi(m)\chi(m)$.
- The characteristic class is the pullback by ϕ of the generator of $H^1(U(1), \mathbb{Z}) = \mathbb{Z}$.
- ϕ and χ are isomorphic if they are equal.
- ϕ and χ are equivalent if and only if they are homotopic.
- The connective structure is included in ϕ.
- The curvature is $\phi^{-1} d\phi$.
- Holonomy on a 0 dimensional submanifold $\{m_0, \ldots, m_r\}$ is the product $\phi(m_0) \ldots \phi(m_r)$.

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- The pullback is pullback of functions $f^*(\phi) = \phi \circ f$.
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Exercise

- Find a definition of $p = -2$ gerbe.
- Show that $\mathbb{Z}$ bundles over M are $p = -1$ gerbes without connective structure. Find a notion of connective structure.

Final remark. For $U(1)$ bundles there is a difference between being trivial and having a trivialisation. A trivialisation is a section of the bundle. Any two trivialisations differ by a function $M \to U(1)$. This is a general feature: we expect trivialisations of a p gerbe to differ by a $p - 1$ gerbe.

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Background

We are interested in surjective submersions $\pi\colon Y \to M$ for which we need some notation. Let $Y^{[p]}$ be the p-fold fibre product of Y with itself. That is

$$Y^{[p]} = \{(y_1, \ldots, y_p) \mid \pi(y_1) = \cdots = \pi(y_p)\} \subset Y^p.$$

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Use the fact that π is a surjective submersion to show that $Y^{[p]} \subset Y^p$ is a submanifold.

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Let $\mathcal{U} = \{U_\alpha \mid \alpha \in I\}$ be an open cover of M and let

$$Y_{\mathcal{U}} = \{(x, \alpha) \mid x \in U_\alpha\} \subset M \times I$$

be the disjoint union of the elements of the cover. Define $\pi \colon Y_{\mathcal{U}} \to M$ by $\pi(x, \alpha) = x$. The fibre product $Y_{\mathcal{U}}^{[p]}$ can be identified with the disjoint union of all the p-fold intersections of elements in $Y_{\mathcal{U}}$. The maps π_i are inclusion maps $U_\alpha \cap U_\beta \to U_\beta$ etc.

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Let $\pi \colon P \to M$ be a principal G bundle for a Lie group G. In this case $P^{[p]} = P \times G^{p-1}$. In particular $P^{[2]} = P \times G$. Later we will need the fact that there is a map $g \colon P^{[2]} \to G$ defined by $p_1 g(p_1, p_2) = p_2$.

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Let $\Omega^q(Y^{[p]})$ be the space of differential q forms on $Y^{[p]}$. Define

$$\delta\colon \Omega^q(Y^{[p-1]}) \to \Omega^q(Y^{[p]}) \quad \text{by} \quad \delta = \sum_{i=1}^{p} (-1)^{p-1}\pi_i^*$$

These maps form the fundamental complex

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Proposition

The fundamental complex is exact for all $q \geq 0$.

Note that if $Y = Y_{\mathcal{U}}$ then this Proposition is a well-known result about the Čech de Rham double complex. See, for example Bott and Tu *Differential Forms in Algebraic Topology*.

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We also need some facts about $U(1)$ bundles. These are well known for hermitian line bundles. If $R \to X$ is a $U(1)$ bundle we can define a new $U(1)$ bundle $R^* \to X$ by letting $R^* = R$ as manifolds but giving it the inverse action so if $r \in R^*$ and $z \in U(1)$ and $\star$ denotes the new action on R^* we have $r \star z = rz^{-1}$. For $\star$ to be a right action depends critically on the fact that $U(1)$ is abelian so that

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Consider Q and R both $U(1)$ bundles over X. Define an equivalence relation on $Q \times R$ by $(q, r) \sim (qw, rw^{-1})$ for any $w \in U(1)$ and let $[q, r]$ denote the equivalence class of (q, r). Denote by $Q \otimes R$ the space of equivalence classes and let $U(1)$ act on it on the right by $[q, r]w = [q, rw]$. Again this requires $U(1)$ to be abelian.

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Consider Q and R both $U(1)$ bundles over X. Define an equivalence relation on $Q \times R$ by $(q, r) \sim (qw, rw^{-1})$ for any $w \in U(1)$ and let $[q, r]$ denote the equivalence class of (q, r). Denote by $Q \otimes R$ the space of equivalence classes and let $U(1)$ act on it on the right by $[q, r]w = [q, rw]$. Again this requires $U(1)$ to be abelian.

If you want to do these constructions with non-abelian bundles you need the group to act on both the left and right so that Q and R are bibundles.

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Exercise

If P is a $U(1)$ bundle show that $P \otimes P^*$ is canonically trivial.

Finally some notation:

If $g\colon Y^{[p-1]} \to U(1)$ we define $\delta(g)\colon Y^{[p]} \to U(1)$

$$\delta(g) = (g \circ \pi_1)(g \circ \pi_2)^{-1}(g \circ \pi_3) \cdots .$$

If $P \to Y^{[p-1]}$ is a $U(1)$ bundle we define a $U(1)$ bundle $\delta(P) \to Y^{[p]}$ by

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Bundle gerbes

(M. J Lon. Math. Soc. (2) 54 (1996))

Definition

A bundle gerbe over M is a pair (P, Y) where $Y \to M$ is a surjective submersion and $P \to Y^{[2]}$ is a $U(1)$ bundle satisfying:

- There is a bundle gerbe multiplication which is a smooth isomorphism

$$m : P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \to P_{(y_1,y_3)}$$

  for all $(y_1, y_2, y_3) \in Y^{[3]}$. Here $P_{(y_1,y_2)}$ denotes the fibre of P over (y_1, y_2).
- This multiplication is associative, that is the following diagram commutes for all $(y_1, y_2, y_3, y_4) \in Y^{[4]}$:

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Note that to be technically correct we should say what smooth means. We can do that by formulating the multiplication as a map of bundles over $Y^{[3]}$ in the form

$$m: \pi_3^*(P) \otimes \pi_1^*(P) \to \pi_2^*(P)$$

but the definition above gives a better idea of what is happening.

We can show using the gerbe multiplication that there are natural isomorphisms $P_{(y_1,y_2)} \cong P^*_{(y_2,y_1)}$ and $P_{(y,y)} \simeq Y^{[2]} \times U(1)$.

Over every point m of M we have a groupoid. The objects are the elements of the fibre Y_m and the morphisms between y_1 and y_2 in Y_m are $P_{(y_1,y_2)}$. Composition comes from the bundle gerbe multiplication. If we call a groupoid a $U(1)$ groupoid if it is transitive and the group morphisms of a point is isomorphic to $U(1)$, then the algebraic conditions on the bundle gerbe are captured by saying it is a bundle of $U(1)$ groupoids.

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If $f\colon N \to M$ then we can pullback $Y \to M$ to $f^*(Y) \to N$ with a map $\hat{f}\colon f^*(Y) \to Y$ covering f. There is an induced map $\hat{f}^{[2]}\colon f^*(Y)^{[2]} \to Y^{[2]}$. Let

$$f^*(P, Y) = (\hat{f}^{[2]*}(P), f^*(Y)).$$

All this is doing is pulling back the $U(1)$ groupoid at $f(n) \in M$ and placing it at $n \in N$.

If (P, Y) is a bundle gerbe then $(P, Y)^* = (P^*, Y)$ is also a bundle gerbe called the dual of (P, Y).

If (P, Y) and (Q, X) are bundle gerbes we can form the fibre product $Y \times_M X \to M$, a new surjective submersion and then define a $U(1)$ bundle

$$P \otimes Q \to (Y \times_M X)^{[2]} = Y^{[2]} \times_M X^{[2]}$$

by

$$(P \otimes Q)_{((y_1,x_1),(y_2,x_2))} = P_{(y_1,y_2)} \otimes Q_{(x_1,x_2)}.$$

We define $(P, Y) \otimes (Q, X) = (P \otimes Q, Y \times_M X)$.

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Show that

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Next we want a connective structure. $P \to Y^{[2]}$ is a $U(1)$ bundle so we can pick a connection A. Call it a bundle gerbe connection if it respects the bundle gerbe multiplication. It follows that the curvature $F_A \in \Omega^2(Y^{[2]})$ satisfies $\delta(F_A) = 0$. From the exactness of the fundamental complex there must be an $f \in \Omega^2(Y)$ such that $F_A = \delta(f)$. As δ commutes with d we have $\delta(df) = d\delta(f) = dF_A = 0$. Hence $df = \pi^*(\omega)$ for some $\omega \in \Omega^3(M)$. So $\pi^*(d\omega) = d\pi^*(\omega) = ddf = 0$ and ω is closed. In fact

$$\left[\frac{1}{2\pi i}\omega\right] = r(DD((P,Y))) \in H^3(M,\mathbb{R}).$$

We call f a curving for A, the pair (A, f) a connective structure for (P, Y) and ω is called the three-curvature of the connective structure (A, f).

The curving f is the B-field in string theory applications.

Why do bundle gerbe connections exist ? Pick any A and consider $\delta(A)$. This is a connection on $\delta(P) \to Y^{[3]}$. The bundle gerbe multiplication defines a section s of $\delta(P)$ and $\delta(A)(s) = \alpha \in \Omega^1(Y^{[3]})$. Moreover $\delta(\alpha) = 0$. Hence $\alpha = \delta(a)$ for some $a \in \Omega^1(Y^{[2]})$ and $\delta(A - a) = 0$. This means that $A - a$ is a bundle gerbe connection.

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Replace $U(1)$ in the definition of bundle gerbe by $\mathbb{Z}$. Show that the result is a theory of $p = 0$ gerbes. Relate it to $U(1)$ bundles.

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Examples

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Let $\mathcal{U}$ be a cover of M and $Y = Y_{\mathcal{U}}$. Then $P \to Y_{\mathcal{U}}^{[2]}$ is a choice of a $U(1)$ bundle $P_{\alpha\beta} \to U_\alpha \cap U_\beta$ for each double overlap and we recover the definition of a gerb due to Hitchin and Chatterjee. (See Chatterjee's thesis: http://www.maths.ox.ac.uk/~hitchin/hitchinstudents/chatterjee.pdf).

If we choose a good cover and local sections of $s_\alpha \colon U_\alpha \to Y$ then the pullback bundles $(s_\alpha, s_\beta)^*(P) \to U_\alpha \cap U_\beta$ also define a Hitchin-Chatterjee gerb. This is the local form of a bundle gerbe.

Notice that for a $U(1)$ bundle (0-gerbe) the local data on double overlaps is a function (-1-gerbe) and for a bundle gerbe (1-gerbe) the local data on double overlaps is a $U(1)$ bundle (0-gerbe). You can extend this idea and consider bundle 2-gerbes whose local data is bundle gerbes on double overlaps (see Stevenson's thesis math.DG/0004117).

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To find further examples of bundle gerbes we look for natural occurrences of three dimensional cohomology. One such place is on a compact, simple Lie group.

Example

(Mickelsson hep-th/0308235). Let $M = SU(n)$. Define

$$Y = \{(X, z) \mid \det(X - z1) \neq 0\} \subset SU(n) \times U(1)$$

with the projection $Y \to SU(n)$. An element of $Y^{[2]}$ is a triple (X, w, z) where w and z are not eigenvalues of X. Let $W_{(X,w,z)}$ be the direct sum of all the eigenspaces of X for eigenvalues between w and z (in an anti-clockwise direction) and let $P_{(X,w,z)}$ be the $U(1)$ frame bundle of $\det(W_{(X,w,z)})$. If w, z and u are in anti-clockwise order around $U(1)$ then

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Example

For the general compact, simple Lie group see Meinrenken `math.DG/0209194`. This bundle gerbe can be given a connective structure for which the holonomy over a three-dimensional submanifold is the Wess-Zumino-Witten term. See Gawedzki, *Rev.Math.Phys.* **14**, *(2002) 1281–1334*.

Example

The Faddeev-Mickelsson bundle gerbe.
This is a bundle gerbe on the space of connections modulo gauge transformations for a bundle on an odd-dimensional manifold. The construction is similar to the group case but with the group elements replaced by Dirac operators coupled to connections. The Dixmier-Douady class is the Faddeev-Mickelsson anomaly. See Carey & M. *Lett. Math. Phys.* **37**, *29–36, (1996)*.

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Another place we can find three cohomology is in bundle lifting problems. Let

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Example

Let PK be the space of all based paths in K a compact, simple Lie group. Let $PK \to K$ be the map which evaluates the path at its end. This is the path-fibration which is an ΩK bundle over K. The lifting bundle gerbe for this gives another realisation of the bundle gerbe over a compact, simple Lie group.

Exercise

Check all the details in the examples above.

Finally we need the notion of bundle gerbe isomorphism. This is the obvious thing. If (P, Y) and (Q, X) are bundle gerbes over M we say they are isomorphic if there is a diffeomorphism $Y \to X$ commuting with the projection. This gives rise to a diffeomorphism $Y^{[2]} \to X^{[2]}$ which is covered by a $U(1)$ bundle isomorphism $P \to Q$ preserving the bundle gerbe multiplication.

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Stable isomorphism

(M & Stevenson J. Lond. Math. Soc. (2), **62**, 925–937, (2000))

So what does it mean if $DD((Y, M)) = 0$? Consider $R \to Y$ a $U(1)$ bundle and $\delta(R) \to Y$ defined by $\delta(R)_{(y_1,y_2)} = R_{y_1} \otimes R^*_{y_2}$. Then

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so $\delta(R)$ has a bundle gerbe product which is associative.

We call a bundle gerbe (P, Y) trivial if there is a $U(1)$ bundle $R \to Y$ with $P \simeq \delta(R)$.

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A bundle gerbe is trivial if and only if its Dixmier-Douady class vanishes.

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So what does it mean if $DD((Y, M)) = 0$? Consider $R \to Y$ a $U(1)$ bundle and $\delta(R) \to Y$ defined by $\delta(R)_{(y_1,y_2)} = R_{y_1} \otimes R^*_{y_2}$. Then

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so $\delta(R)$ has a bundle gerbe product which is associative.

We call a bundle gerbe (P, Y) trivial if there is a $U(1)$ bundle $R \to Y$ with $P \simeq \delta(R)$.

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Proposition

(P, Y) and (X, Q) have the same Dixmier-Douady class if and only if $(P, Y) \otimes (Q, X)^*$ is trivial.

Proof.

$(P, Y) \otimes (Q, X)^*$ is trivial if and only if
$0 = DD((P, Y) \otimes (Q, X)^*) = DD((P, Y)) - DD((Q, X))$. □

We say that (P, Y) and (Q, X) are stably isomorphic if $(P, Y)^* \otimes (Q, X)$ is trivial. A choice of a trivialisation is called a stable isomorphism. It can be shown that stable isomorphisms can be composed and that bundle gerbes, with stable isomorphisms form a two-category. (See Stevenson's thesis `math.DG/0004117`). Stable isomorphism is the correct notion of equivalence for bundle gerbes. The name comes from the fact that (P, Y) and (Q, X) are stably isomorphic if and only if there are trivial gerbes (T_1, Z_1) and (T_2, Z_2) such that

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Stable isomorphism gives a convenient way of understanding the relationship between a bundle gerbe and its local data arising from an open cover and sections of Y. Let (P, Y) be a bundle gerbe and $X \to M$ a surjective submersion. Assume there is a map of surjective submersions $\phi\colon X \to Y$ covering the projections to X. Then there is an induced map $\phi^{[2]}\colon X^{[2]]} \to Y^{[2]}$ and we can pull back P to make a new bundle gerbe $(\phi^{[2]*}(P), X)$.

Exercise

Show that $(\phi^{[2]*}(P), X)$ is stably isomorphic to (P, Y). You can find an explicit trivialisation.

If $\mathcal{U}$ is a local cover and $s_\alpha\colon U_\alpha \to Y$ local sections we can define $s\colon Y_{\mathcal{U}} \to Y$ by $s(x, \alpha) = s_\alpha(x)$. The local $U(1)$ bundles $P_{\alpha\beta}\colon U_\alpha \cap U_\beta$ arise by pulling back P with $s^{[2]}$. Hence the gerb you get by this process is stably isomorphic to the original bundle gerbe.

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If $\mathcal{U}$ is a local cover and $s_\alpha\colon U_\alpha \to Y$ local sections we can define $s\colon Y_{\mathcal{U}} \to Y$ by $s(x, \alpha) = s_\alpha(x)$. The local $U(1)$ bundles $P_{\alpha\beta}\colon U_\alpha \cap U_\beta$ arise by pulling back P with $s^{[2]}$. Hence the gerb you get by this process is stably isomorphic to the original bundle gerbe.

Stable isomorphism gives a convenient way of understanding the relationship between a bundle gerbe and its local data arising from an open cover and sections of Y. Let (P, Y) be a bundle gerbe and $X \to M$ a surjective submersion. Assume there is a map of surjective submersions $\phi\colon X \to Y$ covering the projections to X. Then there is an induced map $\phi^{[2]}\colon X^{[2]]} \to Y^{[2]}$ and we can pull back P to make a new bundle gerbe $({\phi^{[2]}}^*(P), X)$.

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The final property we wanted a 1-gerbe to have is that its connective structure should give rise to holonomy over surfaces. Let (P, Y) be a bundle gerbe with connective structure (A, f) over M and let $\Sigma \subset M$ be a surface. By pulling back to Σ we may as well assume $\Sigma = M$. Because $H^3(\Sigma, \mathbb{Z}) = 0$ we know that there is a trivialisation $R \to Y$. Using the fundamental complex we can show that R admits a connection α with $\delta(\alpha) = A$ under the isomorphism $\delta(R) = P$. If F_α is the curvature of α we must have $\delta(F_\alpha) = F_A = \delta(f)$. Hence $\delta(f - F_\alpha) = 0$ so $f - F_\alpha = \pi^*(\mu)$ for $\mu \in \Omega^2(\Sigma)$. We define the holonomy of (A, f) over Σ by

$$\mathrm{hol}(\Sigma, (A, f)) = \exp\left(\int_\Sigma \mu\right).$$

Exercise

Check that the definition of holonomy is independent of all choices.
Show that if $\Sigma = \partial Y$ and ω is the three-curvature then

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Example

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We will use Gajer's results to obtain a classifying theory for bundle gerbes.

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Let (P, Y) be a bundle gerbe on M. Choose a map $f: M \to B^2U(1)$ such that $f^*[u] = DD((P, Y))$ where u is the generator of $H^3(B^2U(1), \mathbb{Z})$. We have $EBU(1) \to BBU(1)$ which is a $BU(1)$ bundle. We also have a central extension

$$0 \to U(1) \to EU(1) \to BU(1) \to 0$$

and hence a lifting bundle gerbe on $B^2U(1)$. This lifting bundle gerbe has Dixmier-Douady class u so its pullback under f has Dixmier-Douady class $f^*(u)$ and hence is stably isomorphic to (P, Y).

Note that the pullback of a lifting bundle gerbe is the lifting bundle gerbe of the pullback bundle. Hence every bundle gerbe is stably isomorphic to a lifting bundle gerbe.

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Gerbes

So what about gerbes as in Brylinski's book *Loop spaces, Characteristic Classes and Geometric Quantization*? We can turn a bundle gerbe (P, Y) over M into a sheaf of groupoids as follows. Let $U \subset M$ be open and define a groupoid $\mathcal{G}(U)$ with objects sections of Y over U. The morphisms between two sections s and t are the sections of the $U(1)$ bundle $(s,t)^*(P) \to U$ where $(s,t)\colon U \to P^{[2]}$. Composition of morphisms can be done pointwise with the bundle gerbe multiplication.

Of course sections may only exist for 'small enough' open sets so we will need to sheafify the objects and also the morphisms. The final outcome will be a sheaf of groupoids with band the sheaf of smooth maps into $U(1)$.

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Transgression

In the case of a $U(1)$ bundle $P \to M$ with connection there are only two kinds of holonomy. Either we can compute holonomy around a loop to get an element of $U(1)$ or we can compute holonomy (parallel transport) along a path γ to get an element in $P^*_{\gamma(0)} \otimes P_{\gamma(1)}$.

With surfaces we have more interesting options. We have seen that a bundle gerbe (P, Y) with connective structure on M defines holonomy on closed surfaces in M. What about surfaces with boundary? The bundle gerbe can be used to define a $U(1)$ bundle $\hat{P} \to LM$ on the loop space of M. This is a geometric version of the fact that we can take the Dixmier-Douady class of the bundle gerbe on M and use the evaluation map

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If Σ is a surface with positively oriented boundary components $\gamma_1, \ldots, \gamma_r$ then holonomy along Σ is an element of

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If some of the boundary components γ_i are negatively oriented we dualise the corresponding $\hat{P}_{\gamma_i}$. This is related to Segal's notion of a 'string connection' *Phil. Trans. Roy. Soc. Lond. A*, **359**, *Number 1784, 1389–1398, (2001).*

Other groups

We could replace $U(1)$ by any abelian group A and the Dixmier-Douady class would be in $H^2(M, A)$. For example if $A = \mathbb{Z}_2$ there is the lifting bundle gerbe for

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Bundle two-gerbes

A bundle two-gerbe is a pair $(\mathcal{P}, Y)$ where $Y \to M$ is a submersion and $\mathcal{P} \Rightarrow P^{[2]}$ is a bundle gerbe. There is a multiplication and some other conditions. The characteristic class of a bundle two-gerbe is a four class so we look for naturally occuring four classes. One place is the Pontrjagin class of a G bundle $P \to M$. might expect to find a bundle two-gerbe associated to any G bundle $P \to M$ whose characteristic class is the Pontrjagin class of the bundle P. To construct this we consider the map $g \colon P^{[2]} \to G$ and pull back the bundle gerbe on G to $P^{[2]}$. The holonomy of this bundle two-gerbe over a three-dimensional submanifold is the exponential of the integral of the Chern-Simons term over the submanifold. See Brylinski's book, Stevenson's thesis `math.DG/0004117` and *Proc. Lond. Math. Soc. (3)* **88**, *405–435 (2004)*.

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