Toric Geometry of $G_2$-manifolds

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Outline

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Symplectic geometry
HyperKähler geometry

$G_2$ Manifolds
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The Delzant Picture

**(Compact symplectic toric manifolds)** $\to$ **Delzant polytopes**

$(M^{2n}, \omega)$ symplectic with a *Hamiltonian* action of $G = T^n$:
*moment map* $G$-invariant $\mu : M \to g^* \cong \mathbb{R}^n$ with

$$d\langle \mu, X \rangle = X \cdot \omega \quad \forall X \in g.$$  

- $b_1(M) = 0 \implies$ each symplectic $T^n$-action is Hamiltonian
- $\dim(M/T^n)$ equals dimension of target space of $\mu$
- image is Delzant polytope
  
  $$\mu(M) = \Delta = \{ a \in \mathbb{R}^n \mid \langle a, u_k \rangle \leq \lambda_k, \ k = 1, \ldots, m \}$$
- stabiliser of any point is a (connected) subtorus of dimension $n - \text{rank } d\mu$
**HyperKähler manifolds**

\((M, g, \omega_I, \omega_J, \omega_K)\) is hyperKähler if each \((g, \omega_A = g(A \cdot, \cdot))\) is Kähler and \(IJ = K = -JI\)

Then \(\dim M = 4n\) and \(g\) is Ricci-flat, holonomy in \(Sp(n) \leq SU(2n)\)

Ricci-flatness implies:

*if \(M\) is compact, then any Killing vector field is parallel so the holonomy of \(M\) reduces*

So take \((M, g)\) non-compact and complete instead

Swann (2016) and Dancer and Swann (2017), following Bielawski (1999), Bielawski and Dancer (2000), Goto (1994) and Anderson et al. (1989)
Hypertoric is complete hyperKähler $M^{4n}$ with tri-Hamiltonian $G = T^n$ action: have $G$-invariant map (hyperKähler moment map)

$$\mu = (\mu_I, \mu_J, \mu_K): M \to \mathbb{R}^3 \otimes g^* \quad d\langle \mu_A, X \rangle = X \lrcorner \omega_A$$

- $\dim(M/T^n)$ is $3n$, the dimension of target space of $\mu$
- stabiliser of any point is a (connected) subtorus of dimension $n - \frac{1}{3} \text{rank } d\mu$
- Locally (Lindström and Roček, 1983)

$$g = (V^{-1})_{ij} \theta_i \theta_j + V_{ij}(d\mu_I^i d\mu_I^j + d\mu_J^i d\mu_J^j + d\mu_K^i d\mu_K^j),$$

with $(V_{ij})$ positive-definite, and harmonic on each $a + \mathbb{R}^3 \otimes v$

- $\mu(M) = \mathbb{R}^{3n}$ with configuration of flats (possibly infinitely many) $H(u_k, \lambda_k) = \{ a \in \text{Im } \mathbb{H} \otimes \mathbb{R}^n \mid \langle a, u_k \rangle = \lambda_k \}$
- $n = 1$: $V(p) = c + \sum_{q \in Q \subset \mathbb{R}^3} (2\|p - q\|)^{-1}, c \geq 0, V(p) < +\infty$ at some $p$
**$G_2$ MANIFOLDS**

$M^7$ with $\varphi \in \Omega^3(M)$ pointwise of the form

$$\varphi = e_{123} - e_{145} - e_{167} - e_{246} - e_{275} - e_{347} - e_{356},$$

$$e_{ijk} = e_i \wedge e_j \wedge e_k$$

Specifies metric $g = e^2_1 + \cdots + e^2_7$, orientation $\text{vol} = e_{1234567}$ and four-form

$$*\varphi = e_{4567} - e_{2345} - e_{2367} - e_{3146} - e_{3175} - e_{1256} - e_{1247}$$

via

$$6g(X, Y) \text{vol} = (X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi$$

There is also a cross-product

$$g(X \times Y, Z) = \varphi(X, Y, Z)$$

with $X \times Y \perp X, Y$

Holonomy of $g$ is in $G_2$ when $d\varphi = 0 = d*\varphi$, a parallel $G_2$-structure

Then $g$ is Ricci-flat
**Multi-Hamiltonian actions**

Joint work with Thomas Bruun Madsen

$(M, \alpha)$ manifold with closed $\alpha \in \Omega^p(M)$ preserved by $G = T^n$

This is *multi-Hamiltonian* if it there is a $G$-invariant $\nu: M \to \Lambda^{p-1} g^*$ with

$$d\langle \nu, X_1 \wedge \cdots \wedge X_{p-1} \rangle = \alpha(X_1, \ldots, X_{p-1}, \cdot)$$

for all $X_i \in g$

- take $n > p - 2$
- $\nu$ invariant $\iff$ $\alpha$ pulls-back to 0 on each $T^n$-orbit
- $b_1(M) = 0 \implies$ each $T^n$-action preserving $\alpha$ is multi-Hamiltonian

For $(M, \varphi)$ a parallel $G_2$-structure, can take $\alpha = \varphi$ and/or $\alpha = \ast \varphi$
### Multi-Hamiltonian parallel $G_2$-manifolds

#### Proposition

Suppose $(M, \varphi)$ is a parallel $G_2$-manifold with $T^n$-symmetry multi-Hamiltonian for $\alpha = \varphi$ and/or $\alpha = *\varphi$. Then $2 \leq n \leq 4$.

$q$: dimension of orbit space $M^7/T^n$

$k$: dimension of target of multi-moment map $\Lambda^2\mathbb{R}^n$ and/or $\Lambda^3\mathbb{R}^n$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$q$</th>
<th>$\alpha$</th>
<th>$k$</th>
<th>note</th>
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<td>2</td>
<td>5</td>
<td>$\varphi$</td>
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<td>Madsen and Swann (2012)</td>
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<td>$\varphi$ &amp; *$\varphi$</td>
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</tr>
<tr>
<td>4</td>
<td>3</td>
<td>$\varphi$</td>
<td>6</td>
<td>Baraglia (2010)</td>
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<tr>
<td></td>
<td></td>
<td>*$\varphi$</td>
<td>4</td>
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**Toric $G_2$**

**Definition**

A toric $G_2$ manifold is a parallel $G_2$-structure $(M, \varphi)$ with an action of $T^3$ multi-Hamiltonian for both $\varphi$ and $\ast \varphi$.

Let $U_1, U_2, U_3$ generate the $T^3$-action, then $\varphi(U_1, U_2, U_3) = 0$, with multi-moment maps $(\nu, \mu) = (\nu_1, \nu_2, \nu_3, \mu) : M \to \mathbb{R}^4$

$$d\nu_i = U_j \wedge U_k \downarrow \varphi = (U_j \times U_k)^b \quad (i j k) = (1 2 3)$$

$$d\mu = U_1 \wedge U_2 \wedge U_3 \downarrow \ast \varphi$$

Recall $\varphi = e_{123} - e_{145} - e_{167} - e_{246} - e_{275} - e_{347} - e_{356}$

If $U_i$ are linearly independent at $p$, then there is a $G_2$-basis so that $\text{Span}\{U_1, U_2, U_3\} = \text{Span}\{E_5, E_6, E_7\}$. The repeated cross-products of the $U_i$ then generate $TM$ and $(d\nu, d\mu)$ is of full rank 4, so $(\nu, \mu)$ induces a local diffeomorphism

$$M_0/T^3 \to \mathbb{R}^4$$
The flat model

\[ M = S^1 \times \mathbb{C}^3 \]

Standard flat \( \varphi = \frac{i}{2} dx(dz_{11} + dz_{22} + dz_{33}) + \text{Re}(dz_{123}) \)

Preserved by \( T^3 = S^1 \times T^2 \leq S^1 \times SU(3) \)

Stabilisers \( T^2 \) at \( S^1 \times \{0\} \) and \( T^1 \) at \( S^1 \times (z_i = 0 = z_j, i \neq j) \)

Multi-moment maps

\[ 4(\nu_1 - i\mu) = z_1 z_2 z_3, \quad 4\nu_2 = |z_2|^2 - |z_3|^2, \quad 4\nu_3 = |z_3|^2 - |z_1|^2 \]

Topologically \( M/T^3 = \mathbb{C}^3/T^2 = C(S^5)/T^2 = C(S^5/T^2) = C(S^3) = \mathbb{R}^4 \)

The ring \( P(\mathbb{R}^6)^{T^2} \) of invariant polynomials has basis \( \mu, \nu_1, \nu_2, \nu_3 \) and \( t = |z_3|^2 \). By Schwarz (1975) any smooth invariant function on \( \mathbb{C}^3/T^2 \)

is a smooth function of these five invariant polynomials. However, they satisfy

\[ t(t + 2\nu_2)(t - 2\nu_3) = \nu_1^2 + \mu^2, \quad t \geq \max\{0, -2\nu_2, 2\nu_3\} \quad (S) \]

The linear projection \( (t, \nu, \mu) \mapsto (\nu, \mu) \) is a homeomorphism of this set on to \( \mathbb{R}^4 \)
GENERAL PICTURE

Proposition

All isotropy groups of the $T^3$ action are connected and act on the tangent space as maximal tori in (a) $1 \times SU(3)$, (b) $1_3 \times SU(2)$ or (c) $1_7$

Local tangent space models are flat model around (a) $S^1 \times (0, 0, 0)$ or (b) $S^1 \times (1, 0, 0)$. (b) is the Hopf fibration, topologically rigid.
At (a) (full), $\nu_2$ and $\nu_3$ agree with the flat model to order 3, $\nu_1$ and $\mu$ to order 4. Analysis of the singularity (S) and degree arguments give

Theorem

Let $M$ be a full toric $G_2$-manifold, then $M/T^3$ is homeomorphic to a smooth four-manifold. Moreover, the multi-moment map $(\nu, \mu)$ induces a local homeomorphism $M/T^3 \rightarrow \mathbb{R}^4$.

Configuration data: lines in $(\mu = \text{constant})$ of rational slope. Any intersection is triple, with an integrality condition.
**Smooth behaviour**

$M_0 \to M_0/T^3$ is a principal torus bundle with connection one-forms $\theta_i \in \Omega^1(M_0)$ satisfying $\theta_i(U_j) = \delta_{ij}$, $\theta_i(X) = 0$ $\forall$ $X \perp U_1, U_2, U_3$

On $M_0$, put

$$B = (g(U_i, U_j)) \quad \text{and} \quad V = B^{-1} = \frac{1}{\det B} \adj B$$

**Theorem**

$$g = \frac{1}{\det V} \theta^t \adj(V) \theta + d\nu^t \adj(V) d\nu + \det(V) d\mu^2$$

$$\varphi = -\det(V) d\nu_{123} + d\mu d\nu^t \adj(V) \theta + \bigwedge_{i,j,k} \theta_{ij} d\nu_k$$

$$*\varphi = \theta_{123} d\mu + \frac{1}{2 \det(V)} (d\nu^t \adj(V) \theta)^2 + \det(V) d\mu \bigwedge_{i,j,k} \theta_i d\nu_{jk}$$
Such \((g, \varphi, *\varphi)\) defines a parallel \(G_2\)-structure if and only if
\(V \in C^\infty(M_0/T^3, S^2\mathbb{R}^3)\) is a positive-definite solution to
\[
\sum_{i=1}^{3} \frac{\partial V_{ij}}{\partial \nu_i} = 0 \quad j = 1, 2, 3 \quad \text{(divergence-free)}
\]
and
\[
L(V) + Q(dV) = 0 \quad \text{(elliptic)}
\]
where
\[
L = \frac{\partial^2}{\partial \mu^2} + \sum_{i,j} V_{ij} \frac{\partial^2}{\partial \nu_i \partial \nu_j}
\]
and \(Q\) is a quadratic form with constant coefficients

\(L\) and \(Q\) are preserved up to scale by \(GL(3, \mathbb{R})\) change of basis; this specifies \(Q\) uniquely

**Proposition**

* Solutions $V$ to the divergence-free equation are given locally by $A \in C^\infty(M_0/T^3, S^2\mathbb{R}^3)$ via

\[
V_{ii} = \frac{\partial^2 A_{jj}}{\partial v_k^2} + \frac{\partial^2 A_{kk}}{\partial v_j^2} - 2 \frac{\partial^2 A_{jk}}{\partial v_j \partial v_k}
\]

\[
V_{ij} = \frac{\partial^2 A_{ik}}{\partial v_j \partial v_k} + \frac{\partial^2 A_{jk}}{\partial v_i \partial v_k} - \frac{\partial^2 A_{ij}}{\partial v_k^2} - \frac{\partial^2 A_{kk}}{\partial v_i \partial v_j}
\]

$(i \ j \ k) = (1 \ 2 \ 3)$
**Diagonal solutions**

\[ V = \text{diag}(V_1, V_2, V_3) \text{ (divergence-free) and off-diagonal terms in} \]

(elliptic)

\[ \frac{\partial V_i}{\partial v_i} = 0 \quad \frac{\partial V_i}{\partial v_j} \frac{\partial V_j}{\partial v_i} = 0 \quad (i \neq j) \]

*Either* \( V = \text{diag}(V_1(v_2, \mu), V_2(v_3, \mu), V_3(v_1, \mu)) \) linear in each variable

E.g. \( V = \mu_1^3, \mu > 0, \) full holonomy \( G_2: \)

\[ g = \frac{1}{\mu} (\theta_1^2 + \theta_2^2 + \theta_3^2) + \mu^2 (dv_1^2 + dv_2^2 + dv_3^2) + \mu^3 d\mu^2 \]

\[ d\theta_i = dv_j \wedge dv_k \quad (ijk) = (123) \]

*Or* get elliptic hierarchy \( V_3 = V_3(\mu), V_2 = V_2(v_3, \mu), V_1 = V_1(v_2, v_3, \mu) \)

\[ \frac{\partial^2 V_3}{\partial \mu^2} = 0 \quad \frac{\partial^2 V_2}{\partial \mu^2} + V_3 \frac{\partial^2 V_2}{\partial v_3^2} = 0 \quad \frac{\partial^2 V_1}{\partial \mu^2} + V_2 \frac{\partial^2 V_1}{\partial v_2^2} + V_3 \frac{\partial^2 V_1}{\partial v_3^2} = 0 \]

E.g. \( V_3 = \mu, \quad V_2 = \mu^3 - 3v_3^2, \quad V_1 = 2\mu^5 - 15\mu^2 v_3^2 - 5v_2^2 \)
**COMPLETE EXAMPLES**

The flat model $S^1 \times \mathbb{C}^3$

Bryant and Salamon (1989) metrics and their generalisations by Brandhuber et al. (2001) and Bogoyavlenskaya (2013) on $S^3 \times \mathbb{R}^4$: complete, cohomogeneity one with symmetry group $SU(2) \times SU(2) \times S^1 \times \mathbb{Z}/2 \supset T^3$ — only one-dimensional stabilisers.
References I


References II


**References III**


