Reduced group C^*-algebras of reductive Lie groups

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Abstract

These notes are a summary of the description of the reduced C^*-algebra $C^*_r G$ of a connected linear reductive Lie group G outlined by Wassermann and worked out further by Clare, Crisp and Higson. We give the background information in representation theory needed to understand this description, and then state the results. We also describe the K-theory of $C^*_r G$.

Contents

1 Introduction

Let G be a connected, linear reductive Lie group. Its reduced C^*-algebra $C^*_r G$ is the closure in the operator norm of the algebra of convolution operators on $L^2(G)$ by functions in $L^1(G)$. The algebra $C^*_r G$ contains a lot of information about the the tempered representations of G. If G is semisimple, these are precisely the irreducible representations of G that occur in the Plancherel decomposition of $L^2(G)$. For this reason, it is useful to have an explicit description of $C^*_r G$.

Wassermann gave an outline of such a description in [18]. Part of this description was worked out in more detail by Clare, Crisp and Higson in [5]. In these notes, we give the representation theoretic background needed to understand that description, and state the result. We also discuss the K-theory of $C^*_r G$.

These notes are aimed at readers who are familiar with C^*-algebras and K-theory, but not necessarily with Lie groups and representation theory.

2 Preliminaries

Throughout these notes, G will be a Lie group, with Lie algebra $\mathfrak{g}$. All Lie algebras and Lie groups are assumed to be finite-dimensional. We fix a maximal compact subgroup $K < G$, with Lie algebra $\mathfrak{k}$. We also fix a left Haar measure dg on G. (All Haar measures used will be left invariant.)

2.1 Reductive and semisimple groups

The Lie algebra $\mathfrak{g}$ is *reductive* if for every ideal $\mathfrak{a} \subset \mathfrak{g}$ there is an ideal $\mathfrak{b} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$. It is *simple* if it has no nontrivial ideals, and *semisimple* if the equivalent conditions of Proposition 2.1 hold.

Proposition 2.1. *The following conditions on a finite-dimensional Lie algebra* $\mathfrak{g}$ *are equivalent.*

1. $\mathfrak{g}$ *has no nonzero solvable ideals;*
2. $\mathfrak{g}$ *is a direct sum of simple Lie algebras;*
3. *the Killing form* B *on* $\mathfrak{g}$*, defined by*

$$B(X, Y) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$$

for $X, Y \in \mathfrak{g}$*, is nondegenerate.*

If these conditions hold, $\mathfrak{g}$ *is called* semisimple*.*

Proof. See Theorem 1.42 and 1.51 in [10]. □

A Lie algebra is reductive if and only if it is the direct sum of an abelian and a semisimple Lie algebra. (In particular, semisimple Lie algebras are reductive.)

The group G is called reductive or semisimple if $\mathfrak{g}$ has the corresponding property. We will call G linear if it is a closed subgroup of $GL(n, \mathbb{C})$ for some $n \in \mathbb{N}$.

2.2 Cartan subalgebras and subgroups

Let $\mathfrak{g}$ be a complex Lie algebra. Let $\mathfrak{h} \subset \mathfrak{g}$ be a nilpotent complex subalgebra. For $\alpha \in \mathfrak{h}^*$, set

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g}; \text{for all } Y \in \mathfrak{h} \text{ there is an } n \in \mathbb{N} \text{ such that } (\mathrm{ad}(Y) - \alpha(Y))^n X = 0\}. \tag{2.1}$$

Then one has the decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^* \text{ s.t. } \mathfrak{g}_\alpha \neq 0} \mathfrak{g}_\alpha,$$

and since $\mathfrak{h}$ is nilpotent, $\mathfrak{h} \subset \mathfrak{g}_0$. (See Proposition 2.5 in [10].)

Definition 2.2. The subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a *Cartan subalgebra* if $\mathfrak{h} = \mathfrak{g}_0$. Then the *roots* of $(\mathfrak{g}, \mathfrak{h})$ are the nonzero $\alpha \in \mathfrak{h}^*$ for which $\mathfrak{g}_\alpha \neq 0$. The *root space* associated to a root α is the space $\mathfrak{g}_\alpha$. The *Weyl group* associated to these roots is the subgroup of the orthogonal group of the real span of the roots generated by the reflections in the orthogonal complements of the roots, with respect to some inner product.

Cartan subalgebras of complex Lie algebras are unique up to conjugation.

Theorem 2.3. *If $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are Cartan subalgebras of a complex Lie algebra, then there is a $a \in \mathrm{Int}(\mathfrak{g})$, the analytic subgroup of $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g})$ with Lie algebra $\mathrm{ad}(\mathfrak{g})$, such that*

$$\mathfrak{h}_2 = a(\mathfrak{h}_1).$$

Proof. See Theorem 2.15 in [10]. □

For semisimple Lie algebras, Cartan subalgebras and the associated root spaces have additional properties.

Theorem 2.4. *If $\mathfrak{g}$ is a complex semisimple Lie algebra, then*

- *all Cartan subalgebras are abelian;*
- *a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra if and only if $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$ diagonalises simultaneously and $\mathfrak{h}$ equals the zero weight space in this diagonalisation;*

- *all root spaces are one-dimensional, and one may take* $n = 1$ *in* (2.1).

Proof. See Proposition 2.10, Corollary 2.13 and Proposition 2.21 in [10]. $\square$

Definition 2.5. If $\mathfrak{g}$ is a *real* Lie algebra, then a *Cartan subalgebra* of $\mathfrak{g}$ is a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ whose complexification $\mathfrak{h}_{\mathbb{C}}$ is Cartan subalgebra of the complexification $\mathfrak{g}_{\mathbb{C}}$. For a reductive group G, the *Cartan subgroup* associated to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is the centraliser of $\mathfrak{h}$ in G.

Note that not all Cartan subalgebras of a real Lie algebra need to be conjugate in $\mathfrak{g}$; only their complexifications are conjugate in $\mathfrak{g}_{\mathbb{C}}$. This does imply that all Cartan subalgebras have the same dimension. This dimension is the *rank* of $\mathfrak{g}$.

2.3 The Plancherel theorem

Suppose G is linear, connected and reductive. Let $\hat{G}$ be the unitary dual of G, i.e. the set of all equivalence classes of unitary irreducible representations of G. For $\pi \in \hat{G}$, we denote its representation space by $\mathcal{H}_\pi$. Consider the field of Hilbert spaces

$$\mathcal{E} := \coprod_{\pi \in \hat{G}} \mathcal{H}_\pi \otimes \mathcal{H}_\pi^* \to \hat{G}.$$

Here $\mathcal{H}_\pi \otimes \mathcal{H}_\pi^*$ is the completion in the natural inner product on the algebraic tensor product. So $\mathcal{H}_\pi \otimes \mathcal{H}_\pi^*$ is isomorphic to the Hilbert space of Hilbert–Schmid operators on $\mathcal{H}_\pi$.

For $f \in C_c^\infty(G)$ and $\pi \in \hat{G}$, the operator

$$\pi(f) := \int_G f(g)\pi(g)\, dg$$

on $\mathcal{H}_\pi$ is trace-class, hence Hilbert–Schmid. So $\pi(f) \in \mathcal{H}_\pi \otimes \mathcal{H}_\pi^*$. In this way, we obtain a section $\mathcal{F}(f)$ of $\mathcal{E}$, given by

$$\mathcal{F}(f)(\pi) = \pi(f),$$

for all $\pi \in \hat{G}$. Let μ be the *Plancherel measure* on $\hat{G}$, and let $L^2(\mathcal{E}, \mu)$ be the Hilbert space of square-integrable sections of $\mathcal{E}$ with respect to μ. The Plancherel theorem states that $\mathcal{F}$ extends to a unitary isomorphism

$$\mathcal{F}\colon L^2(G) \xrightarrow{\cong} L^2(\mathcal{E}, \mu). \tag{2.2}$$

See Theorem 13.11 in [9] for this fact and the form of μ.

The map $\mathcal{F}$ is $G \times G$-equivariant, in the following sense. Let L and R be the left and right regular representations of G in $L^2(G)$, respectively. Then for all $f \in C_c^\infty(G)$, $\pi \in \hat{G}$ and $g_1, g_2 \in G$,

$$\mathcal{F}(L(g_1)R(g_2)f)(\pi) = \pi(g_1) \otimes \pi^*(g_2)\mathcal{F}(f)(\pi).$$

In this sense, the isomorphism $\mathcal{F}$ is a decomposition of the representation $L^2(G)$ of $G \times G$ into irreducibles. One usually writes

$$L^2(G) \cong \int_{\hat{G}}^{\oplus} \mathcal{H}_\pi \otimes \mathcal{H}_\pi^* \, d\mu(\pi) := L^2(\mathcal{E}, \mu).$$

Remark 2.6. Unitarity of the map (2.2) is equivalent to the equality

$$h(e) = \int_{\hat{G}} \mathrm{tr}(\pi(h)) \, d\mu(\pi) \tag{2.3}$$

for all $h \in C_c^\infty(G)$. Here $\mathrm{tr}(\pi(h))$ is the global character of π applied to f. Let $f \in C_c^\infty(G)$, and let $f^* \in C_c^\infty(G)$ be given by $f^*(g) = \delta(g^{-1})\bar{f}(g^{-1})$ for all $g \in G$, where δ is the modular function. If $h = f * f^*$, then (2.3) is precisely the equality

$$\|f\|^2_{L^2(G)} = \|\mathcal{F}(f)\|^2_{L^2(\mathcal{E},\mu)}.$$

2.4 Tempered representations

Let π be a unitary representation of G in a Hilbert space $\mathcal{H}$. Let $(-,-)_{\mathcal{H}}$ be the inner product on $\mathcal{H}$. A vector $v \in \mathcal{H}$ is K-*finite* if $\pi(K)v$ spans a finite-dimensional linear subspace of $\mathcal{H}$. A K-*finite matrix coefficient* of π is a function on G of the form

$$m_{v,w} \colon g \mapsto (v, \pi(g)w)_{\mathcal{H}},$$

for K-finite vectors $v, w \in \mathcal{H}$.

Definition 2.7. The representation π is *tempered* if all its K-finite matrix coefficients are in $L^{2+\varepsilon}(G)$, for all $\varepsilon > 0$.

Let $\hat{G}_{\mathrm{temp}} \subset \hat{G}$ be the subset of equivalence classes of tempered irreducible representations. The relevance of tempered representations is that the Plancherel measure is supported in $\hat{G}_{\mathrm{temp}}$. (Again, see Theorem 13.11 in [9] and point (2) at the start of Section VIII.11 in [9] .) So

$$L^2(G) \cong L^2(\mathcal{E}, \mu) = L^2(\mathcal{E}|_{\hat{G}_{\mathrm{temp}}}, \mu) =: \int_{\hat{G}_{\mathrm{temp}}}^{\oplus} \mathcal{H}_\pi \otimes \mathcal{H}_\pi^* \, d\mu(\pi). \tag{2.4}$$

For $f \in L^1(G)$, let the bounded operator $f* -$ on $\mathcal{B}(L^2(G))$ be given by left convolution with f. A central role in the description of group C^*-algebras will be played by the following consequence of the Plancherel theorem.

Corollary 2.8. *For all* $f \in C_c^\infty(G)$, *we have*

$$\|f* -\|_{\mathcal{B}(L^2(G))} = \sup_{\pi \in \hat{G}_{\mathrm{temp}}} \|\pi(f)\|_{\mathcal{B}(\mathcal{H}_\pi)}.$$

Proof. Since the map (2.2) is unitary, we have for all $f \in C_c^\infty(G)$,

$$\|f* -\|_{\mathcal{B}(L^2(G))} = \|\mathcal{F} \circ (f* -) \circ \mathcal{F}^{-1}\|_{\mathcal{B}(L^2(\mathcal{E},\mu))}.$$

And for all $\varphi \in L^2(\mathcal{E}, \mu)$ and $\pi \in \hat{G}$, one computes that

$$(\mathcal{F}(f * \mathcal{F}^{-1}\varphi))(\pi) = (\pi(f) \otimes 1_{\mathcal{H}_\pi^*})\varphi(\pi).$$

In other words,

$$\mathcal{F} \circ (f* -) \circ \mathcal{F}^{-1} = (\pi(f) \otimes 1_{\mathcal{H}_\pi^*})_{\pi \in \hat{G}}.$$

Since μ is supported in $\hat{G}_{\mathrm{temp}}$, one can use φ supported near any given π to deduce that

$$\|\mathcal{F} \circ (f* -) \circ \mathcal{F}^{-1}\|_{\mathcal{B}(L^2(\mathcal{E},\mu))} = \sup_{\pi \in \hat{G}_{\mathrm{temp}}} \|\pi(f)\|_{\mathcal{B}(\mathcal{H}_\pi)}.$$

□

2.5 Reduced group C^*-algebras

Let G be any locally compact group. Its *reduced group* C^**-algebra* is the closure in the operator norm of the algebra

$$\{f* -; f \in L^1(G)\} \subset \mathcal{B}(L^2(G))$$

of convolution operators on $L^2(G)$ by functions in $L^1(G)$. This algebra is isomorphic to the completion of the convolution algebra $L^1(G)$ in the norm

$$\|f\|_{C_r^*G} := \|f* - \|_{\mathcal{B}(L^2(G))},$$

with $*$-operation defined by

$$f^*(g) = \delta(g^{-1})\bar{f}(g^{-1}),$$

for $f \in L^1(G)$ and $g \in G$. Here δ is the modular function on G.

Now suppose that G is connected, linear and semisimple. Then Corollary 2.8 implies that for all $f \in L^1(G)$,

$$\|f\|_{C_r^*G} = \sup_{\pi \in \hat{G}_{\mathrm{temp}}} \|\pi(f)\|_{\mathcal{B}(\mathcal{H}_\pi)}. \tag{2.5}$$

The aim of these notes is to give an explicit description of C_r^*G and its K-theory for such groups, or more generally for connected, linear reductive groups. This involves the notions of discrete series representations, discussed in Section 3 and parabolic induction, discussed in Section 4. Those notions will be used to classify the tempered representations of G, which is an important step in the description of C_r^*G.

3 The discrete series

3.1 Discrete series representations

Suppose G is linear, connected and reductive. We consider a unitary irreducible representation π of G in a Hilbert space $\mathcal{H}$.

Definition 3.1. The representation π belongs to the *discrete series* of G if all its matrix coefficients are in $L^2(G)$.

The set of equivalence classes of discrete series representations of G will be denoted by $\hat{G}_{\mathrm{ds}}$.

Proposition 3.2. *An irreducible unitary representation belongs to the discrete series if and only if it is equivalent to a closed subspace (i.e. a direct summand) of the left regular representation of* G *in* $L^2(G)$.

Proof. See Theorem 8.51(b) in [9]. If the matrix coefficients of a representation π are in $L^2(G)$, an equivariant isometric embedding $B : \mathcal{H} \to L^2(G)$ can be defined as follows. Fix a nonzero $v_0 \in \mathcal{H}$, and define the map B by

$$\big(B(v)\big)(g) = (\pi(g^{-1})v, v_0)_{\mathcal{H}},$$

for $v \in \mathcal{H}$ and $g \in G$. □

Proposition 3.2 implies that the discrete series representations are exactly those with positive Plancherel measure. The term discrete series is also motivated by the fact that the discrete series is a discrete subset of $\hat{G}$ in the *Fell topology*.

Definition 3.3. The *Fell topology* on $\hat{G}$ is defined as follows. The closure of a set $X \subset \hat{G}$ is the set of all $\pi \in \hat{G}$ such that all matrix coefficients of π can be approximated uniformly on compact subsets by matrix coefficients of representations in X.

3.2 Classification of discrete series representations

Suppose that G is linear, connected and semisimple. One has the following explicit criterion for the existence of discrete series representations.

Theorem 3.4. *The group* G *has discrete series representations if and only if* $\mathrm{rank}(G) = \mathrm{rank}(K)$, *i.e.* G *has a compact Cartan subgroup.*

Proof. See Theorem 12.20 in [9]. □

For the non-exceptional simple real Lie groups, this criterion leads to Table 3.2, which was taken from [7].

Now suppose that there is a maximal torus $T < K$ which is a Cartan subgroup of G, so that G has discrete series representations. Let R be the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Let R_c denote the set of compact roots, i.e. those of $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, and let $R_n := R \setminus R_c$ be the set of noncompact roots. Fix an element $\lambda \in i\mathfrak{t}^*$. Suppose λ is nonsingular, in the sense that $(\lambda, \alpha) \neq 0$ for all roots $\alpha \in R$. Let R^+ be the set of positive roots defined by

$$R^+ := \{\alpha \in R; (\alpha, \lambda) > 0\}. \tag{3.1}$$

Let ρ be half the sum of the roots in R^+, and let ρ_c be half the sum of the positive compact roots in $R_c^+ := R^+ \cap R_c$.

Group G	Max. cpt. $K < G$	rank(G)	rank(K)	Discrete series?
$SL(n, \mathbb{C})$	$SU(n)$	$2n-2$	$n-1$	no
$SL(n, \mathbb{R})$	$SO(n)$	$n-1$	$\lfloor \frac{n}{2} \rfloor$	iff $n = 2$
$SL(n, \mathbb{H})$	$Sp^*(n)$	$2n-1$	n	no
$SU(p, q)$	$S(U(p) \times U(q))$	$p+q-1$	$p+q-1$	yes
$SO(n, \mathbb{C})$	$SO(n)$	$2\lfloor \frac{n}{2} \rfloor$	$\lfloor \frac{n}{2} \rfloor$	no
$SO(p, q)$	$S(O(p) \times O(q))$	$\lfloor \frac{p+q}{2} \rfloor$	$\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor$	iff pq even
$O^*(2n)$	$U(n)$	n	n	yes
$Sp(n, \mathbb{C})$	$Sp^*(n)$	$2n$	n	no
$Sp(n, \mathbb{R})$	$U(n)$	n	n	yes
$Sp^*(p, q)$	$Sp^*(p) \times Sp^*(q)$	$p+q$	$p+q$	yes

Table 1: Harish–Chandra's criterion $\mathrm{rank}(G) = \mathrm{rank}(K)$ for the existence of discrete series representations, for the non-exceptional real Lie groups

Theorem 3.5. *If $\lambda + \rho$ is analytically integral, there is a discrete series representation π_λ of G such that*

1. *if $\nu := \lambda + \rho - 2\rho_c$, and π^K_ν is the irreducible representation of K with highest weight ν, then the multiplicity of π^K_ν in $\pi_\lambda|_K$ is one;*
2. *if μ is the highest weight of an irreducible representation of K with nonzero multiplicity in $\pi_\lambda|_K$, then there are nonnegative integers n_α such that*

$$\mu = \nu + \sum_{\alpha \in R^+} n_\alpha \alpha.$$

Two such discrete series representations π_λ and $\pi_{\lambda'}$ are equivalent if and only if there is an element w of the Weyl group of R_c such that $\lambda' = w\lambda$.

Proof. See Theorem 9.20 in [9]. □

In the setting of Theorem 3.5, the element $\lambda \in i\mathfrak{t}^*$ is called the *Harish–Chandra parameter* of π_λ. The representation π^K_ν is the *lowest K-type* of π_λ, and ν is the *Blattner parameter* of π_λ.

Theorem 3.6. *Every discrete series representation of G equals one of the representations π_λ of Theorem 3.5.*

Proof. See Theorem 12.21 in [9]. □

Theorems 3.5 and 3.6 give a complete classification of the discrete series representations of G.

We will need to consider discrete series representations of possibly disconnected groups. See for example [12] for the classification of these.

3.3 Example: $\mathrm{SL}(2, \mathbb{R})$

For any $n \in \mathbb{N}$, consider the semisimple Lie group $G = \mathrm{SL}(n, \mathbb{R})$. Then $K = \mathrm{SO}(n)$ is a maximal subgroup of G. Write $n = 2k$ if n is even, and $n = 2k + 1$ if n is odd. Then a maximal torus in $\mathrm{SO}(n)$ is isomorphic to

$$\underbrace{\mathrm{SO}(2) \times \cdots \times \mathrm{SO}(2)}_{k \text{ factors}}.$$

Hence K has rank k. A Cartan subalgebra of the complexified Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ is formed by the diagonal elements, and has complex dimension $n - 1$. Hence $\mathrm{rank}(G) = n - 1$. By Theorem 3.4, $\mathrm{SL}(n, \mathbb{R})$ therefore has discrete series representations if and only if

- $n = 2k$ is even, and $k = n - 1$; or
- $n = 2k + 1$ is odd, and $k = n - 1$.

In other words, $\mathrm{SL}(n, \mathbb{R})$ has discrete series representations precisely if $n = 2$.

For the rest of this subsection, we consider the group $\mathrm{SL}(2, \mathbb{R})$.

3.3.1 Cartan subgroups

The Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ has two conjugacy classes of Cartan subalgebras. One is represented by $\mathfrak{t} = \mathbb{R}X$, where

$$X := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The other is represented by $\mathfrak{h} = \mathbb{R}Y$, where

$$Y := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The corresponding Cartan subgroups are the compact group

$$T := SO(2),$$

and the noncompact group

$$A := \left\{ \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}; r \neq 0 \right\}.$$

Since we are going to construct discrete series representations of $SL(2, \mathbb{R})$, we focus on the compact Cartan subgroup T.

The corresponding root space decomposition is

$$\mathfrak{sl}(2, \mathbb{C}) = \mathbb{C}X \oplus \mathbb{C}E_\alpha + \mathbb{C}E_{-\alpha},$$

where

$$E_\alpha := \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}; \qquad E_{-\alpha} := \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

One can compute that

$$[X, E_{\pm\alpha}] = \pm 2i E_{\pm\alpha}.$$

Hence the root system of $(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{t}_\mathbb{C})$ is $\{\pm\alpha\}$, with α determined by

$$\alpha(X) = 2i.$$

There are no compact roots, i.e. $R_c = \emptyset$, since $K = T$ is abelian.

3.3.2 Discrete series representations

Let a nonzero element $\lambda \in i\mathfrak{t}^*$ be given. Write $\lambda = l\alpha$, for an $l \in \mathbb{R}$. The choice of positve roots determined by λ is $R^+ = \{\alpha\}$ if $l > 0$, and $R^+ = \{-\alpha\}$ if $l < 0$. Hence

$$\rho = \mathrm{sign}(l) \frac{1}{2} \alpha; \quad \rho_c = 0.$$

For any $a \in \mathbb{R}$, one has

$$\exp(aX) = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}.$$

Hence $\ker \exp = 2\pi\mathbb{Z}X$. Since $\rho(2\pi X) = \mathrm{sign}(l)2\pi i$, we see that ρ is analytically integral. Hence $\lambda + \rho$ is analytically integral if and only if λ is, which is the case precisely if $\lambda(2\pi X) = 4\pi i l \in 2\pi i\mathbb{Z}$, i.e. if

$$\lambda = \lambda_n := \frac{n}{2}\alpha,$$

for a nonzero integer n. The discrete series representations of $SL(2, \mathbb{R})$ are precisely the representations π_λ given in Theorem 3.5, for these values of λ. Write $\pi_n := \pi_{\lambda_n}$. No two of these are equivalent, since the Weyl group of the compact roots is trivial.

4 Induced representations

4.1 Non-unitary induction

Let G be a locally compact topological group, and $H < G$ a closed subgroup. Let $\pi\colon H \to GL(V)$ be a continous representation of H in a topological vector space V. Let $C(G, V)$ be the space of continuous functions on G with values in V. Consider the action by H on $C(G, V)$ given by

$$(hf)(g) = \pi(h)f(gh),$$

for $h \in H$, $f \in C(G, V)$ and $g \in G$. Let $\mathrm{ind}_H^G(V) := C(G, V)^H$ be the fixed point set of this action.

Definition 4.1. The *induced representation* of π from H to G is the representation $\mathrm{ind}_H^G(\pi)$ of G in the vector space $\mathrm{ind}_H^G(V)$ given by

$$(\mathrm{ind}_H^G(\pi)(g)f)(g') = f(g^{-1}g'),$$

for $g, g' \in G$ and $f \in \mathrm{ind}_H^G(V)$.

An issue is that $\mathrm{ind}_H^G(\pi)$ may not be unitarisable, even if π is unitary.

4.2 Unitary induction

Now suppose that G, and hence H, is a Lie group. Suppose V is a Hilbert space, with inner product $(-, -)_V$ (complex-linear in the first entry). Then one can slightly modify the definition of induced representations so that the induced representation is unitary (or unitarisable) if π is.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. The adjoint representation $\mathrm{Ad}\colon G \to GL(\mathfrak{g})$ restricts and projects to

$$\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\colon H \to GL(\mathfrak{g}/\mathfrak{h}).$$

Consider the function $\delta\colon H \to \mathbb{R}$ given by

$$\delta(h) = |\det(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h))|^{-1},$$

fro all $h \in H$. For any homomorphism $\chi\colon H \to \mathbb{C}^{\times}$, we write $\mathbb{C}_{\chi}$ for the complex numbers with the representation of H defined by χ.

Lemma 4.2. *For any $\alpha \in \mathbb{C}$, the line bundle*

$$G \times_H \mathbb{C}_{\delta^{\alpha}} \to G/H$$

is the bundle of α-densities on G/H.

By this lemma, every element of $C(G, \mathbb{C}_{\delta})^H$ defines a continuous density on G/H. Consider the sesquilinear map

$$(-,-)\colon V \otimes \mathbb{C}_{\delta^{1/2}} \times V \otimes \mathbb{C}_{\delta^{1/2}} \to \mathbb{C}_{\delta} \tag{4.1}$$

given by

$$(v_1 \otimes \lambda_1, v_2 \otimes \lambda_2) = (v_1, v_2)_V \lambda_1 \bar{\lambda}_2,$$

for $v_1, v_2 \in V$ and $\lambda_1, \lambda_2 \in \mathbb{C}_{\delta^{1/2}}$. Now suppose that π is unitary. Then for all $\varphi, \psi \in C(G, V \otimes \mathbb{C}_{\delta^{1/2}})^H$, the function (φ, ψ) on G mapping $g \in G$ to $(\varphi(g), \psi(g))$ is in $C(G, \mathbb{C}_{\delta})^H$. Hence it defines a density on G/H, so it can be integrated. For $\varphi, \psi \in C_c(G, V \otimes \mathbb{C}_{\delta^{1/2}})^H$, we define

$$(\varphi, \psi)_{L^2} := \int_{G/H} (\varphi, \psi).$$

Let $\mathrm{Ind}_H^G(V)$ be the completion of $C_c(G, V \otimes \mathbb{C}_{\delta^{1/2}})^H$ in this inner product.

Definition 4.3. The *unitarily induced representation* of π from H to G is the representation $\mathrm{Ind}_H^G(\pi)$ of G in $\mathrm{Ind}_H^G(V)$, given by

$$(\mathrm{Ind}_H^G(\pi)(g)f)(g') = f(g^{-1}g'),$$

for $g, g' \in G$ and $f \in \mathrm{Ind}_H^G(V)$.

Lemma 4.4. *If π is unitary, then so is* $\mathrm{Ind}_H^G(\pi)$.

If G/H is compact, as it is if H is a parabolic subgroup, then $\mathrm{Ind}_H^G(V)$ is the completion of $\mathrm{ind}_H^G(V \otimes \mathbb{C}_{\delta^{1/2}})$ in the inner product (4.1).

4.3 Cuspidal parabolic subgroups

Suppose G is linear, connected and reductive. Let $\mathfrak{s} \subset \mathfrak{g}$ be an $\mathrm{Ad}(K)$-invariant subspace such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra such that

$$\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{s}).$$

(Any Cartan subalgebra is conjugate to one with this property.) Set

- $H := Z_G(\mathfrak{h})$;
- $\mathfrak{a} := \mathfrak{h} \cap \mathfrak{s}$;
- $A :=$ the analytic subgroup of G with Lie algebra $\mathfrak{a}$;
- $\mathfrak{m} :=$ the orthogonal complement to $\mathfrak{a}$ in $Z_{\mathfrak{g}}(\mathfrak{a})$;
- $M_0 :=$ the analytic subgroup of G with Lie algebra $\mathfrak{m}$;
- $M := Z_K(\mathfrak{a})M_0$.

The subgroup M may be disconnected.

For $\beta \in \mathfrak{a}^*$, set

$$\mathfrak{g}_\beta := \{X \in \mathfrak{g}; \text{for all } Y \in \mathfrak{a}, [Y, X] = \langle \beta, Y \rangle X\}.$$

Consider the restricted root system

$$\Sigma := \Sigma(\mathfrak{g}, \mathfrak{a}) := \{\beta \in \mathfrak{a}^* \setminus \{0\}; \mathfrak{g}_\beta \neq \{0\}\}.$$

Fix a positive system $\Sigma^+ \subset \Sigma$. Consider the nilpotent subalgebra

$$\mathfrak{n} := \bigoplus_{\beta \in \Sigma^+} \mathfrak{g}_\beta.$$

of $\mathfrak{g}$. Let N be the analytic subgroup of G with Lie algebra $\mathfrak{n}$.

Definition 4.5. The *cuspidal parabolic subgroup* of G associated to $\mathfrak{h}$ and Σ^+ is $P := MAN$.

Let $T < K$ be a maximal torus. Then $\mathfrak{t} \cap \mathfrak{m} \subset \mathfrak{m}$ is a Cartan subalgebra, so M has discrete series and limits of discrete series representations. There is a more general notion of parabolic subgroups $P = MAN$. Such a group is called *cuspidal* if M has discrete series representations. All cuspidal parabolic subgroups (up to conjugacy) arise as described in this subsection.

4.4 Parabolic induction

Now suppose that G is reductive, and $H = P = MAN$ is a cuspidal[1] parabolic subgroup. Let $\sigma\colon M \to U(V_\sigma)$ be a unitary representation of M and $\nu \in \mathfrak{a}^*$, so that $e^{i\nu}$ is a one-dimensional unitary representation of A. Let $\rho \in \mathfrak{a}^*$ be half the sum of the positive restricted roots corresponding to N. Let 1_N be the trivial representation of N; then we have the unitary representation $\sigma \otimes e^{i\nu} \otimes 1_N$ of P, in the space $V_\sigma \otimes \mathbb{C}_{e^{i\nu}}$. Let $K < G$ be maximal compact. Let $\rho \in \mathfrak{a}^*$ be half the sum of the elements of Σ^+.

Lemma 4.6. *The space* $\mathrm{Ind}_P^G(V_\sigma \otimes \mathbb{C}_{e^{i\nu}})$ *is the completion of the space*

$$\big\{f \in C(G, V_\sigma);$$
$$\textit{for all } g \in G,\ m \in M,\ a \in A \textit{ and } n \in N,\ f(gman) = e^{-i\nu+\rho}(a)\sigma(m)^{-1}f(g)\big\}$$

in the inner product

$$(f_1, f_2)_{L^2} = \int_K (f_1(k), f_2(k))_{V_\sigma}\, dk, \tag{4.2}$$

where dk *is the Haar measure on* K *such that* $\mathrm{vol}(K) = 1$.

Proof. In this setting, the space $\mathfrak{g}/\mathfrak{h} = \mathfrak{g}/\mathfrak{p}$ identifies with the direct sum of the negative restricted root spaces. Hence $\delta^{1/2}|_A = e^{-\rho}$, while $\delta^{1/2}|_M = 1$ and $\delta^{1/2}|_N = 1$. Also G/P identifies with a quotient of K, since $G = KP$. See Subsection 2.1 in [2] for more details. ☐

The realisation of $\mathrm{Ind}_P^G(V_\sigma \otimes \mathbb{C}_{e^{i\nu}})$ in this lemma is called the *induced picture*.

By Lemma 4.4, the representation $\mathrm{Ind}_P^G(\sigma \otimes e^{i\nu} \otimes 1_N)$ is unitary. Furthermore, $\mathrm{Ind}_P^G(\sigma \otimes e^{i\nu} \otimes 1_N)$ is tempered is σ is. (See Proposition 7.14 and page 198 of [9].)

There are two other realisations of the representation $\mathrm{Ind}_P^G(\sigma \otimes e^{i\nu} \otimes 1_N)$. First, via restriction to K, we have a G-equivariant unitary isomorphism from the space in Lemma 4.6 onto the completion of the space

$$\{f \in C(K, V_\sigma); \text{for all } k \in K,\ m \in K \cap M,\ f(km) = \sigma(m)^{-1}f(k)\},$$

[1]What follows is true for general parabolic subgroups, but we will only apply it to cuspidal ones.

in the inner product (4.2). The action by G on this space is defined as follows. For any $g \in G$, write $g = \kappa(g)\mu(g)a(g)n$, for $\kappa(g) \in K$, $\mu(g) \in M$, $a(g) \in A$ and $n \in N$. Then for all $g \in G$, $k \in K$, and f in the space just defined,

$$(gf)(k) = e^{-(i\nu+\rho)}(a(g))\sigma(\mu(g^{-1}k))^{-1}f(\kappa(g^{-1}k)).$$

The advantage if this realisation is that the representation space does not depend on $\nu \in \mathfrak{a}^*$. This is the *compact picture*.

In the *noncompact picture* of $\mathrm{Ind}_P^G(\sigma \otimes e^{i\nu} \otimes 1_N)$ is the restriction of the space in Lemma 4.6 to $\bar{N}$. The representation space is then independent of σ. It is the space of L^2-functions on $\bar{N}$ with respect to the measure $e^{2\,\mathrm{Re}(\nu)}(a(\bar{n}))d\bar{n}$, with h as above. For any $g \in G$, write $g = \bar{n}(g)\mu(g)a(g)n$, for $\bar{n}(g) \in \bar{N}$, $\mu(g) \in M$, $a(g) \in A$ and $n \in N$. Then for all $g \in G$, $\bar{n} \in \bar{N}$, and $f \in L^2(\bar{N}, e^{2\,\mathrm{Re}(\nu)}(a(\bar{n}))d\bar{n})$,

$$(gf)(\bar{n}) = e^{-(i\nu+\rho)}(a(g^{-1}\bar{n}))\sigma(\mu(g^{-1}\bar{n}))^{-1}f(\bar{n}(g^{-1}\bar{n})).$$

If $P = MAN$ is a minimal parabolic subgroup, then M is compact, so P is cuspidal.

Definition 4.7. The *principal series* of G is the set of representations

$$\left\{P^{\sigma,i\nu} := \mathrm{Ind}_P^G(\sigma \otimes \mathbb{C}_{e^{i\nu}} \otimes 1_N); \sigma \in \hat{M}_{ds}, \nu \in \mathfrak{a}^*\right\},$$

for a minimal parabolic $P = MAN < G$.

4.5 Parabolic induction via Hilbert C^*-modules

(The contents of this subsection are not used in the rest of these notes.)

We continue using the notation and assumptions from the previous section. Write $L := MA$. Fix left Haar measures dg, dl and dn on G, L and N, respectively. Let $d(gN)$ be the G-invariant measure on G/N such that for all $f \in C_c(G)$,

$$\int_G f(g)\, dg = \int_{G/N} \int_N f(gn)\, dn\, d(gN).$$

We define a Hilbert C^*_rL-module $C^*_r(G/N)$ as follows. (See [4].) Consider the space $C_c^\infty(G/N)$. We have a right action on this space by the convolution algebra $C_c^\infty(L)$, given by

$$(f_{G/N} \cdot f_L)(gN) = \int_L \delta^{-1/2}(l) f_{G/N}(gl^{-1}N) f_L(l)\, dl,$$

for all $f_{G/N} \in C_c^\infty(G/N)$, $f_L \in C_c^\infty(L)$ and $g \in G$. Here δ is the same function as before, for $\mathfrak{h} = \mathfrak{p}$, so $\mathfrak{g}/\mathfrak{h} = \bar{\mathfrak{n}}$, the direct sum of the negative restricted root spaces. Consider the $C_c^\infty(L)$-valued inner product on $C_c^\infty(G/N)$ given by

$$(f_{G/N}, f'_{G/N})_{C_c^\infty(L)}(l) = \delta^{-1/2}(l) \int_{G/N} f_{G/N}(gl^{-1}N)\bar{f}'_{G/N}(gn)\, d(gN).$$

for all $f_{G/N}, f'_{G/N} \in C_c^\infty(G/N)$, and $l \in L$. By Proposition 1 in [4], we can use these structures to complete $C_c^\infty(G/N)$ to a right Hilbert $C_r^*(L)$-module $C_r^*(G/N)$.

The action by the convolution algebra $C_c^\infty(G)$ on $C_c^\infty(G/N)$ given by

$$(f \cdot f_{G/N})(gN) = \int_G f(g')f_{G/N}(g'^{-1}gN)\, dg',$$

for $f \in C_c^\infty(G)$ and $f_{G/N} \in C_c^\infty(G/N)$, extends to an action by C_r^*G on $C_r^*(G/N)$ by adjointable operators.

Proposition 4.8. *Let $\sigma\colon M \to U(V_\sigma)$ be a unitary representation of M, and $\nu \in i\mathfrak{a}^*$. There is a unitary isomorphism of Hilbert sopaces*

$$C_r^*(G/N) \otimes_{C_r^*L} (V_\sigma \otimes \mathbb{C}_{e^{i\nu}}) \cong \mathrm{Ind}_P^G(V_\sigma \otimes \mathbb{C}_{e^{i\nu}})$$

intertwining the given $*$-representations of C_r^*G.

This is Corollary 1 in [4].

4.6 Intertwining operators

Let $P = MAN < G$ be a parabolic subgroup, $\sigma \in \hat{M}$ and $\nu \in \mathfrak{a}^*$. Let $w \in N_K(MA)$. Then conjugation by w preserves M and A. So composing σ with conjugation by w, we obtain another representation $w\sigma \in \hat{M}_{ds}$. Furthermore, w acts on $\mathfrak{a}_\mathbb{C}$ and $\mathfrak{a}_\mathbb{C}^*$ via the adjoint action. For any $\nu \in \mathfrak{a}_\mathbb{C}^*$, the induced representation $\mathrm{Ind}_P^G(\sigma \otimes e^\nu \otimes 1_N)$ is still defined even though e^ν is not unitary, for example as the representation in Lemma 4.6 with $i\nu$ replaced by ν. For $\nu \in \mathfrak{a}_\mathbb{C}^*$, define

$$A_P(w, \sigma, \nu)\colon\ \mathrm{Ind}_P^G(V_\sigma \otimes \mathbb{C}_{e^\nu}) \to \mathrm{Ind}_P^G(V_{w\sigma} \otimes \mathbb{C}_{e^{w\nu}})$$

by

$$(A_P(w,\sigma,\nu)f)(g) = \int_{\bar{N}\cap w^{-1}Nw} f(gw\bar{n})\, d\bar{n}$$

for $f \in \mathrm{Ind}_P^G(V_\sigma \otimes \mathbb{C}_{e^{i\nu}})$ and $g \in G$. This integral converges if $\mathrm{Re}(\nu)$ is large enough, in the appropriate sense. If σ is tempered, then the integral converges if f is K-finite and $\mathrm{Re}(\nu)$ lies in the open positive Weyl chamber defined by Σ^+, see Theorem 7.22 and page 198 in [9]. (The convergence issue is especially subtle if M is noncompact.) For arbitrary, in particular imaginary, ν, we use analytic continuation. The map $A_P(w,\sigma,\nu)$ is a unitary intertwining operator. See Subsection VII.4 of [9] and Proposition 8.5 in [11].

Example 4.9. Let $G = SL(2,\mathbb{R})$. Let σ_+ be the trivial representation of $M = \{\pm I_2\}$, and σ_- the nontrivial representation. Let $W = \mathbb{Z}/2\mathbb{Z}$ be the Weyl group of $(SL(2,\mathbb{R}), SO(2))$. Let $w \in W$ be the nontrivial element. For $\nu \in \mathbb{R}$, consider the principal series representation

$$P^{\pm,i\nu} = \mathrm{Ind}_P^G(\sigma_\pm \otimes e^{i\nu} \otimes 1_N).$$

In the noncompact picture (see Subsection 4.4), we have for all $x \in \mathbb{R}$,

$$(A_P(w,\sigma_+,i\nu)f)(x) = \lim_{t\downarrow 0} \int_{\mathbb{R}} \frac{f(x-y)}{|y|^{1-i\nu-t}}\, dy;$$

and

$$(A_P(w,\sigma_-,i\nu)f)(x) = \lim_{t\downarrow 0} \int_{\mathbb{R}} \frac{f(x-y)\,\mathrm{sgn}(y)}{|y|^{1-i\nu-t}}\, dy.$$

See (7.13) in [9].

4.7 The classification of tempered representations

Discrete series representations and parabolic induction can be used to classify tempered representations.

Theorem 4.10. *Every irreducible tempered representation π of G can be obtained by induction from a cuspidal parabolic subgroup $P = MAN < G$ as*

$$\pi = \mathrm{Ind}_P^G(\sigma \otimes e^{i\nu} \otimes 1),$$

where $\nu \in \mathfrak{a}^*$, and σ is a discrete series representation of M, or a limit of discrete series representations *of M.*

Proof. See Theorem 14.76 in [9]. □

Knapp and Zuckerman also determined which P, σ and ν occur in Theorem 4.10, completing the classification of tempered representations. See Theorem 14.2 in [13].

The definition of limits of the discrete series is given in Section XII.7 of [9]. However, even if one needs a limit of the discrete series in the setting of Theorem 4.10, then π is still *contained* in a representation induced from a discrete series representation of M.

Theorem 4.11. *If* $P = MAN$ *is a cuspidal parabolic subgroup,* $\nu \in i\mathfrak{a}^*$*, and* σ *is a limit of discrete series representations of* M*, then there are a parabolic subgroup* $P' = M'A'N' \subset G$*, a discrete series representation* σ' *of* M' *and a* $\nu' \in i\mathfrak{a}'^*$*, such that* $\pi := \mathrm{Ind}_P^G(\sigma \otimes e^{i\nu} \otimes 1)$ *is contained in* $\pi' := \mathrm{Ind}_{P'}^G(\sigma' \otimes e^{\nu'} \otimes 1)$*, in the sense that the global character of* π *equals the sum of the global character of* π' *plus another global character.*

Proof. See Corollary 14.72 in [9]. □

Combining Theorems 4.10 and 4.11, see also Corollary 8.8 in [12], we obtain the following result.

Corollary 4.12. *Every tempered representation of* G *is contained in a representation of the form* $\mathrm{Ind}_P^G(\sigma \otimes e^{i\nu} \otimes 1_N)$ *for a cuspidal parabolic subgroup* $P = MAN < G$*, a discrete series representation* $\sigma \in \hat{M}_{ds}$ *and* $\nu \in \mathfrak{a}^*$*.*

5 Group C^*-algebras

5.1 Bundles of compact operators

Let $P = MAN < G$ be a cuspidal parabolic subgroup, and let $\sigma \in \hat{M}_{ds}$. Consider the bundle of Hilbert spaces $\mathcal{E}_\sigma \to \hat{A}$ whose fibre at $e^{i\nu} \in \hat{A}$ is $\mathrm{Ind}_P^G(V_\sigma \otimes \mathbb{C}_{e^{i\nu}})$. Using the compact picture of parabolic induction (see Subsection 4.4), we can realise $\mathcal{E}_\sigma$ as a trivial bundle. This defines the topology on $\mathcal{E}_\sigma$. Let $\Gamma_0(\mathcal{E}_\sigma)$ be the Hilbert $C_0(\hat{A})$-module of continuous sections of $\mathcal{E}_\sigma$ vanishing at infinity. Consider the bundle of C^*-algebras $\mathcal{K}(\mathcal{E}_\sigma) \to \hat{A}$ whose fibre at $e^{i\nu} \in \hat{A}$ is $\mathcal{K}(\mathrm{Ind}_P^G(V_\sigma \otimes \mathbb{C}_{e^{i\nu}}))$. Again, we topologise this bundle by identifying it with a trivial bundle in the compact picture. Let

$\Gamma_0(\mathcal{K}(\mathcal{E}_\sigma))$ be the C^*-algebra of continuous sections of $\mathcal{K}(\mathcal{E}_\sigma)$ vanishing at infinity.

Consider the map

$$\psi_\sigma \colon L^1(G) \to \Gamma_0(\mathcal{K}(\mathcal{E}_\sigma))$$

given by

$$(\psi_\sigma(f))(e^{i\nu}) = \mathrm{Ind}_P^G(\sigma \otimes e^{i\nu} \otimes 1_N)(f) \quad \in \mathcal{K}(\mathrm{Ind}_P^G(V_\sigma \otimes \mathbb{C}_{e^{i\nu}})),$$

for $f \in L^1(G)$ and $\nu \in \mathfrak{a}^*$. This map extends to a $*$-homomorphism

$$\psi_\sigma \colon C_r^*(G) \to \Gamma_0(\mathcal{K}(\mathcal{E}_\sigma)).$$

Let $\mathrm{Pds}(G)$ be a set of pairs (P, σ), where P runs over a set of representatives of conjugacy classes of cuspidal parabolic subgroups $P = MAN$, and $\sigma \in \widehat{M}_{\mathrm{ds}}$.

Consider the C^*-algebraic direct sum

$$\bigoplus_{(P,\sigma)\in \mathrm{Pds}(G)} \Gamma_0(\mathcal{K}(\mathcal{E}_\sigma)) = \Big\{ T = (T_{P,\sigma})_{P,\sigma \in \mathrm{Pds}(G)} \in \prod_{(P,\sigma)\in \mathrm{Pds}(G)} \Gamma_0(\mathcal{K}(\mathcal{E}_\sigma)); \text{for all } P,\ \lim_{\sigma\to\infty} \|T_{P,\sigma}\| = 0 \Big\}.$$

The maps ψ_σ assemble into a $*$-homomorphism

$$\psi \colon C_r^*(G) \to \bigoplus_{(P,\sigma)\in \mathrm{Pds}(G)} \Gamma_0(\mathcal{K}(\mathcal{E}_\sigma)).$$

This allows us to decompose C_r^*G.

Lemma 5.1. *The map ψ is injective.*

Proof. Let $x \in \ker(\psi)$. Let π be a tempered representation of G. By Corollary 4.12, π is equivalent to a subrepresentation of $\mathrm{Ind}_P^G(\sigma \otimes e^{i\nu} \otimes 1_N)$ for a pair $(P = MAN, \sigma) \in \mathrm{Pds}(G)$ and $e^{i\nu} \in \hat{A}$. Now

$$\mathrm{Ind}_P^G(\sigma \otimes e^{i\nu} \otimes 1_N)(x) = \psi_\sigma(x)(e^{i\nu}) = 0.$$

So $\pi(x) = 0$. So (2.5) implies that $\|x\|_{C_r^*G} = 0$. □

Proposition 5.2. *The image of ψ is*

$$\mathrm{im}(\psi) = \bigoplus_{(P,\sigma)\in \mathrm{Pds}(G)} \mathrm{im}(\psi_\sigma).$$

This is Proposition 5.15 in [5].

5.2 A decomposition of $C^*_r G$

Because of Lemma 5.1 and Proposition 5.2, an explicit description of the images of the maps ψ_σ yields an explicit description of $C^*_r G$. Fix $(P = MAN, \sigma) \in \mathrm{Pds}(G)$. Consider the finite group

$$W := N_K(\mathfrak{a})/Z_K(\mathfrak{a}).$$

Since $N_K(\mathfrak{a}) < N_K(M)$, this group acts on M by conjugation. Hence it acts on $\hat{M}_{\mathrm{ds}}$ via pullbacks by conjugation. Consider the stabiliser

$$W_\sigma := \{w \in W; w\sigma = \sigma\}.$$

For each $w \in W_\sigma$ and $e^{i\nu} \in \hat{A}$, we have the unitary intertwining operator

$$A_P(w, \sigma, \nu)\colon\ \mathrm{Ind}_P^G(V_\sigma \otimes \mathbb{C}_{e^{i\nu}}) \to \mathrm{Ind}_P^G(V_\sigma \otimes \mathbb{C}_{e^{iw\nu}})$$

as in Subsection 4.6. Define the map

$$U_w\colon \Gamma_0(\mathcal{K}(\mathcal{E}_\sigma)) \to \Gamma_0(\mathcal{K}(\mathcal{E}_\sigma))$$

by

$$(U_w T)(e^{i\nu}) = A_P(w, \sigma, \nu) T(e^{iw^{-1}\nu}) A_P(w^{-1}, \sigma, \nu).$$

for all $T \in \Gamma_0(\mathcal{K}(\mathcal{E}_\sigma))$ and $e^{i\nu} \in \hat{A}$.

Consider the subalgebra

$$\Gamma_0(\mathcal{K}(\mathcal{E}_\sigma))^{W_\sigma} := \{T \in \Gamma_0(\mathcal{K}(\mathcal{E}_\sigma)); \text{for all } w \in W_\sigma,\ U_w T = T\}.$$

Theorem 5.3. *The image of* ψ_σ *is*

$$\Gamma_0(\mathcal{K}(\mathcal{E}_\sigma))^{W_\sigma}.$$

See Proposition 6.7 in [5].

Corollary 5.4. *The map* ψ *defines an isomorphism*

$$C^*_r G \cong \bigoplus_{(P,\sigma)\in \mathrm{Pds}(G)} \Gamma_0(\mathcal{K}(\mathcal{E}_\sigma))^{W_\sigma}.$$

This follows from Lemma 5.1, Proposition 5.2 and Theorem 5.3.

5.3 R-groups

To compute the K-theory of $C^*_r G$, we give a more explicit description of the C^*-algebras $\Gamma_0(\mathcal{K}(\mathcal{E}_\sigma))^{W_\sigma}$, up to stable isomorphism.

Definition 5.5. Let $\sigma \in \hat{M}$. Consider the group

$$W'_\sigma := \{w \in W_\sigma; A_P(w, \sigma, 0) \in \mathbb{C}I\}.$$

Theorem 5.6. *If* $\sigma \in \hat{M}_{\mathrm{ds}}$, *then*

$$W_\sigma = W'_\sigma \rtimes R_\sigma$$

for a subgroup $R_\sigma < W_\sigma$ *isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^l$ *for some* $l \in \mathbb{Z}_{\geq 0}$.

See [11, 18].

Theorem 5.7. *The dimension of the algebra of* G*-equivariant operators on* $\mathrm{Ind}_P^G(V_\sigma \otimes \mathbb{C}_1)$ *is the number of elements of* R_σ.

This is Theorem 13.4(iv) in [11]. In particular, this theorem implies that R_σ is trivial (i.e. $W_\sigma = W'_\sigma$) if and only if $\mathrm{Ind}_P^G(V_\sigma \otimes \mathbb{C}_1)$ is irreducible. More generally, suppose that $\mathrm{Ind}_P^G(V_\sigma \otimes \mathbb{C}_1)$ is the direct sum of inequivalent irreducible representations $\pi_1, \ldots, \pi_s$, with multiplicities $m_1, \ldots, m_s$. Then by Schur's lemma, the above theorem states that

$$\sum_{j=1}^{s} m_j^2 = \#R_\sigma. \tag{5.1}$$

Proposition 5.8. *We have a stable isomorphism*

$$\Gamma_0(\mathcal{K}(\mathcal{E}_\sigma))^{W_\sigma} \cong C_0(\mathfrak{a}^*/W'_\sigma) \rtimes R_\sigma.$$

See Corollary 7 and Theorem 9 in [18].

Combining Corollary 5.4 and Proposition 5.8, we obtain the following description of $C^*_r G$ up to stable isomorphism.

Theorem 5.9. *We have a stable isomorphism*

$$C^*_r G \cong \bigoplus_{(P,\sigma) \in \mathrm{Pds}(G)} C_0(\mathfrak{a}^*/W'_\sigma) \rtimes R_\sigma.$$

5.4 The K-theory of $C^*_r G$

Lemma 5.10. *If W'_σ is nontrivial, then* $K_*(C_0(\mathfrak{a}^*/W'_\sigma) \rtimes R_\sigma) = 0$.

See Lemma 10 in [18].

Let $P_{max} = M_{max}A_{max}N_{max} < G$ be the cuspidal parabolic subgroup corresponding to a maximally compact Cartan subgroup. Set $l := \dim(A) - \dim(A_{max})$. (Note that A_{max} has the *lowest* dimension among all factors A of cuspidal parabolic subgroups $MAN < G$.)

Lemma 5.11. *If W'_σ is trivial, then*

$$R_\sigma \cong (\mathbb{Z}/2\mathbb{Z})^l.$$

The vector space $\mathfrak{a}^*$ is $(\mathbb{Z}/2\mathbb{Z})^l$-equivariantly isomorphic to $\mathbb{R}^{\dim(\mathfrak{a})}$, where $(\mathbb{Z}/2\mathbb{Z})^l$ acts on $\mathbb{R}^{\dim(\mathfrak{a})}$ via reflections in the first l entries. So

$$C_0(\mathfrak{a}^*/W'_\sigma) \rtimes R_\sigma \cong \left(C_0(\mathbb{R}^l) \rtimes (\mathbb{Z}/2\mathbb{Z})^l\right) \otimes C_0(\mathbb{R}^{\dim(A_{max})}).$$

See Lemma 12 in [18].

Because of this lemma. we have, if W'_σ is trivial,

$$K_j(C_0(\mathfrak{a}^*/W'_\sigma) \rtimes R_\sigma) = K_{j+\dim(A_{max})}((C_0(\mathbb{R}) \rtimes (\mathbb{Z}/2\mathbb{Z}))^{\otimes l})$$

Now $K_0(C_0(\mathbb{R}) \rtimes (\mathbb{Z}/2\mathbb{Z})) = \mathbb{Z}$ and $K_1(C_0(\mathbb{R}) \rtimes (\mathbb{Z}/2\mathbb{Z})) = 0$, see Lemma 14 in [18]. Also, $\dim(A_{max}) \equiv \dim(G/K) \mod 2$. So by the above arguments and the Künneth theorem (see for example Theorem 23.1.3 in [3]), we have, if W'_σ is trivial,

$$\begin{aligned} K_{\dim(G/K)}(C_0(\mathfrak{a}^*/W'_\sigma) \rtimes R_\sigma) &= \mathbb{Z}; \\ K_{\dim(G/K)+1}(C_0(\mathfrak{a}^*/W'_\sigma) \rtimes R_\sigma) &= 0. \end{aligned}$$

Let $b(P,\sigma) \in K_{\dim(G/K)}(C_0(\mathfrak{a}^*/W'_\sigma) \rtimes R_\sigma)$ be a generator.

Combining this with Theorem 5.9 and Lemma 5.10, we reach the following conclusion.

Theorem 5.12. *We have*

$$K_{\dim(G/K)}(C^*_r G) = \bigoplus_{(P,\sigma)\in \mathrm{Pds}(G); W'_\sigma = \{e\}} \mathbb{Z} b(P,\sigma).$$

and $K_{\dim(G/K)+1}(C^*_r G) = 0$.

By Theorem 5.7 and (5.1), we have $W'_\sigma = \{e\}$ if and only if

$$\sum_{j=1}^{s} m_j^2 = \#W_\sigma,$$

where $m_1, \ldots, m_s$ are the multiplicities of s inequivalent irreducible representations into which $\mathrm{Ind}_P^G(\sigma \otimes \mathbb{C}_1)$ decomposes. If P is a minimal parabolic, then by Theorem 7.2 in [9], the number of irreducible components of $\mathrm{Ind}_P^G(\sigma \otimes \mathbb{C}_1)$ is at most equal to $\#W_\sigma$. Hence in that case, we have $W'_\sigma = \{e\}$ if and only if $\mathrm{Ind}_P^G(\sigma \otimes \mathbb{C}_1)$ is maximally reducible.

5.5 Example: discrete series classes

Suppose G has discrete series representations. Then G is a cuspidal parabolic subgroup of itself (corresponding to a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}$). For this parabolic subgroup, we have $A = \{e\}$, and the group $W = K/K$ is trivial. So by Corollary 5.4, $C^*_r G$ has the subalgebra

$$\bigoplus_{\sigma \in \hat{G}_{ds}} \mathcal{K}(V_\sigma).$$

Hence $K_*(C^*_r G) = K_0(C^*_r G)$ has the subgroup

$$\bigoplus_{\sigma \in \hat{G}_{ds}} \mathbb{Z} b(G, \sigma),$$

where $b(G, \sigma)$ is a generator of $K_0(\mathcal{K}(V_\sigma)) = \mathbb{Z}$. This generator can be described explicitly.

Fix $\sigma \in \hat{G}_{ds}$. Let $x \in V_\sigma$ be a unit vector. Then the matrix coefficient $m_{x,x}$ is in $L^2(G)$. Set

$$d_\sigma := \|m_{x,x}\|_{L^2(G)}^{-2}.$$

This is the formal degree of σ, which is its Plancherel measure. Now $d_\sigma \overline{m_{x,x}}$ is an idempotent in $C^*_r G$, and its class is $b(G, \sigma)$ (up to a sign).

We see that the discrete series embeds into $K_0(C^*_r G)$ without loss of information. This is exploited in [14] to prove results about the discrete series. In [8], this is used to deduce Harish–Chandra's character formula for the discrete series from a fixed point formula in index theory.

5.6 Example: the reduced C^*-algebra of $SL(2,\mathbb{R})$

Let $G = SL(2,\mathbb{R})$. Let σ_+ be the trivial representation of $M = \{\pm I_2\}$, and σ_- the nontrivial representation. Now $W = \mathbb{Z}/2\mathbb{Z}$. $W_{\sigma_\pm} = W$. Let $w \in W$ be the nontrivial element. For $\nu \in \mathbb{R}$, consider the principal series representation

$$P^{\pm,i\nu} = \mathrm{Ind}_P^G(\sigma_\pm \otimes e^{i\nu} \otimes 1_N).$$

We have $-i\nu = wi\nu$ for all $\nu \in \mathbb{R} \cong \mathfrak{a}^*$, and $w\sigma_\pm = \sigma_\pm$ since $M < Z(G)$. Consider the operators

$$A_P(w, \sigma_\pm, i\nu)\colon P^{\pm,i\nu} \to P^{\pm,-i\nu}$$

as in Example 4.9.

The summand $\Gamma_0(\mathcal{K}(\mathcal{E}_{\sigma_+}))^{W_{\sigma_+}}$ can now be identified via the following special case of Proposition 5.8.

Lemma 5.13. *If* $P = MAN < G$ *is a cuspidal parabolic, and* $\sigma \in \hat{M}_{ds}$ *is such that* $\mathrm{Ind}_P^G(\sigma \otimes \mathbb{C}_1)$ *is irreducible, then*

$$\Gamma_0(\mathcal{K}(\mathcal{E}_\sigma))^{W_\sigma} \cong C_0(\mathfrak{a}^*/W_\sigma) \otimes \mathcal{K}.$$

Here $\mathcal{K}$ is the algebra of compact operators on the fibre of the trivial bundle $\mathcal{E}_\sigma$. Note that in the setting of this lemma, we have $W'_\sigma = W_\sigma$. Since $P^{+,0}$ is irreducible, it follows that

$$\Gamma_0(\mathcal{K}(\mathcal{E}_{\sigma_+}))^{W_{\sigma_+}} \cong C_0([0,\infty)) \otimes \mathcal{K}. \tag{5.2}$$

Lemma 5.14. *We have*

$$\Gamma_0(\mathcal{K}(\mathcal{E}_{\sigma_-}))^{W_{\sigma_-}} \cong \big(C_0(\mathbb{R}) \rtimes (\mathbb{Z}/2\mathbb{Z})\big) \otimes \mathcal{K}.$$

Proof. For any $\nu \in \mathbb{R}$, the representation space of $P^{-,i\nu}$ can be identified with $L^2(\mathbb{R})$. For $\nu = 0$, this representation decomposes into the two limits of discrete series $\pi_0^\pm$. Let $V_{\pi_0^\pm}$ be the representation space of $\pi_0^\pm$. Then $V_{\pi_0^+}$ is the space of holomorphic functions f on the upper half plane for which

$$\sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|\, dy$$

is finite. And $V_{\pi_0^-}$ is the space of all complex conjugates of functions in $V_{\pi_0^+}$. (See section 2.4 of [9] for details about these representations.) These spaces embed into $L^2(\mathbb{R})$ by continous extension and restriction to the real line. So the complex conjugation operator C on $L^2(\mathbb{R})$ interchanges $V_{\pi_0^+}$ and $V_{\pi_0^-}$.

It follows from the explicit formula for $A_P(w, \sigma_-, i\nu)$ in Example 4.9 that for all $\nu \in \mathbb{R}$,

$$A_P(w, \sigma_-, -i\nu) = CA_P(w, \sigma_-, i\nu)C.$$

Since C interchanges the subspaces $V_{\pi_0^+}$ and $V_{\pi_0^-}$ of $L^2(\mathbb{R})$ (which are only G-invariant if $\nu = 0$), the claim now follows from Lemma 5.15 below. $\square$

Lemma 5.15. *Consider the action by $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{R}$ such that the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ corresponds minus the identity on $\mathbb{R}$. Let*

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We have a $*$-isomorphism

$$\Phi\colon C_0(\mathbb{R}) \rtimes (\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cong} \big\{a \in C_0(\mathbb{R}) \otimes M_2(\mathbb{C}); \textit{for all } x \in \mathbb{R},\ a(-x) = Ca(x)C\big\},$$

given by

$$\Phi(f)(x) = \begin{pmatrix} f(0,x) & f(1,x) \\ f(1,-x) & f(0,-x) \end{pmatrix}$$

for $f \in C_0(\mathbb{Z}/2\mathbb{Z} \times \mathbb{R})$ and $x \in \mathbb{R}$, where $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$.

See Proposition 2.52 in [19].

Combining (5.2) with Lemma 5.14 and with the arguments from Subsection 5.5, we find that

$$C_r^* \mathrm{SL}(2,\mathbb{R}) \cong \big(C_0([0,\infty)) \otimes \mathcal{K}\big) \oplus \Big(\big(C_0(\mathbb{R}) \rtimes (\mathbb{Z}/2\mathbb{Z})\big) \otimes \mathcal{K}\Big) \oplus \bigoplus_{n \in \mathbb{Z}; n \neq 0} \mathcal{K}(V_{\pi_n}),$$

with π_n as in Subsection 3.3. So $K_0(C_r^* \mathrm{SL}(2,\mathbb{R}))$ is free abelian, with one generator for every nonzero integer n, generating $K_0(\mathcal{K}(V_{\pi_n}))$, and one additional generator, generating $K_0(\big(C_0(\mathbb{R}) \rtimes (\mathbb{Z}/2\mathbb{Z})\big) \otimes \mathcal{K})$. And $K_1(C_r^* \mathrm{SL}(2,\mathbb{R})) = 0$.

5.7 Example: complex groups

Let G be a complex semisimple Lie group. Then all parabolic subgroups of G are conjugate to the minimal parabolic $P = MAN$. For that group, we have $M = \mathbb{T}^n$ and $A \cong \mathbb{R}^n$ for an $n \in \mathbb{N}$. So $\hat{M}_{ds} = \hat{M} = \mathbb{Z}^n$. The principal series representations

$$P^{\sigma,i\nu} = \mathrm{Ind}_P^G(\sigma \otimes e^{i\nu} \otimes 1_N)$$

for $\sigma \in \hat{M}_{ds}$ and $\nu \in \mathfrak{a}^*$, are all irreducible, and exhaust the irreducible tempered representations of G. Two principal series representations $P^{\sigma,i\nu}$ and $P^{\sigma',i\nu'}$ are equivalent if and only if there is a $w \in W$ such that $w\sigma = \sigma'$ and $w\nu = \nu'$. Hence

$$\hat{G}_{\mathrm{temp}} = (\mathbb{Z}^n \times \mathbb{R}^n)/W.$$

By Lemma 5.13, we have

$$\Gamma_0(\mathcal{K}(\mathcal{E}_\sigma))^{W_\sigma} = C_0(\mathfrak{a}^*/W_\sigma) \otimes \mathcal{K}.$$

By Corollary 5.4, we conclude that

$$C_r^*G = \bigoplus_{\sigma \in \hat{M}_{ds}} C_0(\mathfrak{a}^*/W_\sigma) \otimes \mathcal{K} \cong C_0(\hat{G}_{\mathrm{temp}}) \otimes \mathcal{K}.$$

Hence

$$K_j(C_r^*G) = K^j(\hat{G}_{\mathrm{temp}}) = \bigoplus_{\sigma \in \hat{M}_{ds}; W_\sigma = \{e\}} K^j(\mathbb{R}^n).$$

Since $n = \dim A \cong \dim(G/K) \mod 2$, we find that

$$K_{\dim(G/K)}(C_r^*G) = \bigoplus_{\sigma \in \hat{M}_{ds}; W_\sigma = \{e\}} \mathbb{Z},$$

and $K_{\dim(G/K)+1}(C_r^*G) = 0$. See [17] for more details.

5.8 Dirac induction

In [1, 16] discrete series representations are realised geometrically in the L^2-kernel of a Dirac operator on G/K, defined as follows. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be a Cartan decomposition. Consider the inner product on $\mathfrak{s}$ given by the restriction of the Killing form. The adjoint representation

$$\mathrm{Ad}\colon K \to GL(\mathfrak{s})$$

of K on $\mathfrak{s}$ takes values in $SO(\mathfrak{s})$, because the Killing form is $Ad(K)$-invariant, and K is connected. We suppose that it has a lift $\widetilde{Ad}$ to the double cover $Spin(\mathfrak{s})$ of $SO(\mathfrak{s})$. It may be necessary to replace G and K by double covers for this lift to exist; we indicate how to handle the general case below. Then the homogeneous space G/K has a G-equivariant Spin-structure

$$P^{G/K} := G \times_K Spin(\mathfrak{s}) \to G/K.$$

Here $G \times_K Spin(\mathfrak{s})$ is the quotient of $G \times Spin(\mathfrak{s})$ by the action of K defined by

$$k(g, a) = (gk^{-1}, \widetilde{Ad}(k)a),$$

for $k \in K$, $g \in G$ and $a \in Spin(\mathfrak{s})$.

Set $d := \dim(\mathfrak{s}) = \dim(G/K)$. Fix an orthonormal basis $\{X_1, \ldots, X_d\}$ of $\mathfrak{s}$. Let $\Delta_{\mathfrak{s}}$ be the canonical representation of $Spin(d)$. Consider the G-vector bundle

$$E_V := G \times_K (\Delta_{\mathfrak{s}} \otimes V) \to G/K.$$

Note that

$$\Gamma^{\infty}(G/K, E_V) \cong \left(C^{\infty}(G) \otimes \Delta_{\mathfrak{s}} \otimes V\right)^K, \tag{5.3}$$

where K acts on $C^{\infty}(G) \otimes \Delta_{\mathfrak{s}} \otimes V$ by

$$k \cdot (f \otimes \delta \otimes v) = (f \circ l_{k^{-1}} \otimes \widetilde{Ad}(k)\delta \otimes k \cdot v) \tag{5.4}$$

for all $k \in K$, $f \in C^{\infty}(G)$, $\delta \in \Delta_{\mathfrak{s}}$ and $v \in V$. Here $l_{k^{-1}}$ denotes left multiplication by k^{-1}.

Using the basis $\{X_1, \ldots, X_d\}$ of $\mathfrak{s}$ and the isomorphism (5.3), define the differential operator

$$D^V \colon \Gamma^{\infty}(E_V) \to \Gamma^{\infty}(E_V) \tag{5.5}$$

by the formula

$$D^V := \sum_{j=1}^{d} X_j \otimes c(X_j) \otimes 1_V. \tag{5.6}$$

Here in the first factor, X_j is viewed as a left invariant vector field on G, and in the second factor, $c\colon \mathfrak{s} \to End(\Delta_{\mathfrak{s}})$ is the Clifford action. The operator (5.5) is the Spin-Dirac operator on G/K (see e.g. [16], Proposition 1.1.

Let V be an irreducible representation of K. Lafforgue (see also Wassermann [18]) uses the Dirac operator D^V defined in (5.6) to define a *Dirac induction* map

$$\text{D-Ind}_K^G : R(K) \to K_*(C^*_r(G)) \tag{5.7}$$

by

$$\text{D-Ind}_K^G[V] := \left[\left(C_r^*(G) \otimes \Delta_{\mathfrak{s}} \otimes V\right)^K, b\left(D^V\right)\right], \tag{5.8}$$

where $b\colon \mathbb{R} \to \mathbb{R}$ is a normalising function, e.g. $b(x) = \frac{x}{\sqrt{1+x^2}}$. The expression on the right hand side defines a class in Kasparov's KK-group $KK_*(\mathbb{C}, C_r^*(G))$, which is isomorphic to the K-theory group $K_*(C_r^*(G))$. If G/K is even-dimensional, $\Delta_{\mathfrak{s}}$ splits into two irreducible subrepresentations $\Delta_{\mathfrak{s}}^+$ and $\Delta_{\mathfrak{s}}^-$. Then D^V is odd with respect to the grading on $\Delta_{\mathfrak{s}}$, and Dirac induction takes values in $K_0(C_r^*G)$. If G/K is odd-dimensional, then it takes values in $K_1(C_r^*G)$.

If the lift $\widetilde{\mathrm{Ad}}$ does not exist, one uses a double cover $\tilde{K}$ of K, and takes the irreducible representations V of $\tilde{K}$ for which $\Delta_{\mathfrak{s}} \otimes V$ descends to a representation of K.

In [18], Wassermann outlined a proof the Connes–Kasparov conjecture, which states that this Dirac induction map is a bijection, for linear reductive groups. This is based on Theorem 5.12. Lafforgue gave a different proof for semisimple Lie groups, in [15]. The latter proof is not based on the explicit structure of C_r^*G as in Theorem 5.12. Lafforgue's result was generalised to general almost connected Lie groups in [6].

One of the strengths of using $K^*(C_r^*G)$ is that Dirac induction always defines a nonzero element of this group, even though the L^2-kernel of D^V is zero in many cases.

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