Chapter 6

Manifolds, Tangent Spaces, Cotangent Spaces, Vector Fields, Flow, Integral Curves

6.1 Manifolds

In a previous Chapter we defined the notion of a manifold embedded in some ambient space, $\mathbb{R}^N$.

In order to maximize the range of applications of the theory of manifolds it is necessary to generalize the concept of a manifold to spaces that are not a priori embedded in some $\mathbb{R}^N$.

The basic idea is still that, whatever a manifold is, it is a topological space that can be covered by a collection of open subsets, U_α, where each U_α is isomorphic to some "standard model", *e.g.*, some open subset of Euclidean space, $\mathbb{R}^n$.

Of course, manifolds would be very dull without functions defined on them and between them.

This is a general fact learned from experience: Geometry arises not just from spaces but from spaces and interesting classes of functions between them.

In particular, we still would like to "do calculus" on our manifold and have good notions of curves, tangent vectors, differential forms, etc.

The small drawback with the more general approach is that the definition of a tangent vector is more abstract.

We can still define the notion of a curve on a manifold, but such a curve does not live in any given $\mathbb{R}^n$, so it it not possible to define tangent vectors in a simple-minded way using derivatives.

Instead, we have to resort to the notion of chart. This is not such a strange idea.

For example, a geography atlas gives a set of maps of various portions of the earh and this provides a very good description of what the earth is, without actually imagining the earth embedded in 3-space.

Given $\mathbb{R}^n$, recall that the projection functions, $pr_i\colon \mathbb{R}^n \to \mathbb{R}$, are defined by

$$pr_i(x_1, \ldots, x_n) = x_i, \quad 1 \leq i \leq n.$$

For technical reasons, from now on, all topological spaces under consideration will be assumed to be Hausdorff and second-countable (which means that the topology has a countable basis).

Definition 6.1.1 Given a topological space, M, a *chart* (or *local coordinate map*) is a pair, (U, φ), where U is an open subset of M and $\varphi: U \to \Omega$ is a homeomorphism onto an open subset, $\Omega = \varphi(U)$, of $\mathbb{R}^{n_\varphi}$ (for some $n_\varphi \geq 1$).

For any $p \in M$, a chart, (U, φ), is a *chart at p* iff $p \in U$. If (U, φ) is a chart, then the functions $x_i = pr_i \circ \varphi$ are called *local coordinates* and for every $p \in U$, the tuple $(x_1(p), \ldots, x_n(p))$ is the set of *coordinates of p* w.r.t. the chart.

The inverse, (Ω, φ^{-1}), of a chart is called a *local parametrization*.

Given any two charts, (U_1, φ_1) and (U_2, φ_2), if $U_1 \cap U_2 \neq \emptyset$, we have the *transition maps*, $\varphi_i^j: \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ and $\varphi_j^i: \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$, defined by

$$\varphi_i^j = \varphi_j \circ \varphi_i^{-1} \quad \text{and} \quad \varphi_j^i = \varphi_i \circ \varphi_j^{-1}.$$

Clearly, $\varphi_j^i = (\varphi_i^j)^{-1}$.

Observe that the transition maps φ_i^j (resp. φ_j^i) are maps between *open subsets of* $\mathbb{R}^n$.

This is good news! Indeed, the whole arsenal of calculus is available for functions on $\mathbb{R}^n$, and we will be able to promote many of these results to manifolds by imposing suitable conditions on transition functions.

Definition 6.1.2 Given a topological space, M, and any two integers, $n \geq 1$ and $k \geq 1$, a C^k *n-atlas* (or *n-atlas of class* C^k), $\mathcal{A}$, is a family of charts, $\{(U_i, \varphi_i)\}$, such that

(1) $\varphi_i(U_i) \subseteq \mathbb{R}^n$ for all i;

(2) The U_i cover M, i.e.,

$$M = \bigcup_i U_i;$$

(3) Whenever $U_i \cap U_j \neq \emptyset$, the transition map φ_i^j (and φ_j^i) is a C^k-diffeomorphism.

We must insure that we have enough charts in order to carry out our program of generalizing calculus on $\mathbb{R}^n$ to manifolds.

For this, we must be able to add new charts whenever necessary, provided that they are consistent with the previous charts in an existing atlas.

Technically, given a C^k n-atlas, $\mathcal{A}$, on M, for any other chart, (U, φ), we say that (U, φ) is *compatible* with the altas $\mathcal{A}$ iff every map $\varphi_i \circ \varphi^{-1}$ and $\varphi \circ \varphi_i^{-1}$ is C^k (whenever $U \cap U_i \neq \emptyset$).

Two atlases $\mathcal{A}$ and $\mathcal{A}'$ on M are *compatible* iff every chart of one is compatible with the other atlas.

This is equivalent to saying that the union of the two atlases is still an atlas.

It is immediately verified that compatibility induces an equivalence relation on C^k n-atlases on M.

In fact, given an atlas, $\mathcal{A}$, for M, the collection, $\widetilde{\mathcal{A}}$, of all charts compatible with $\mathcal{A}$ is a maximal atlas in the equivalence class of charts compatible with $\mathcal{A}$.

Definition 6.1.3 Given any two integers, $n \geq 1$ and $k \geq 1$, a C^k*-manifold of dimension* n consists of a topological space, M, together with an equivalence class, $\overline{\mathcal{A}}$, of C^k n-atlases, on M. Any atlas, $\mathcal{A}$, in the equivalence class $\overline{\mathcal{A}}$ is called a *differentiable structure of class* C^k *(and dimension* n*) on* M. We say that M is *modeled on* $\mathbb{R}^n$. When $k = \infty$, we say that M is a *smooth manifold*.

Remark: It might have been better to use the terminology *abstract manifold* rather than manifold, to emphasize the fact that the space M is not a priori a subspace of $\mathbb{R}^N$, for some suitable N.

We can allow $k = 0$ in the above definitions. Condition (3) in Definition 6.1.2 is void, since a C^0-diffeomorphism is just a homeomorphism, but φ_i^j is always a homeomorphism.

In this case, M is called a *topological manifold of dimension n*.

We do not require a manifold to be connected but we require all the components to have the same dimension, n.

Actually, on every connected component of M, it can be shown that the dimension, n_φ, of the range of every chart is the same. This is quite easy to show if $k \geq 1$ but for $k = 0$, this requires a deep theorem of Brouwer.

What happens if $n = 0$? In this case, every one-point subset of M is open, so every subset of M is open, i.e., M is any (countable if we assume M to be second-countable) set with the discrete topology!

Observe that since $\mathbb{R}^n$ is locally compact and locally connected, so is every manifold.

Remark: In some cases, M does not come with a topology in an obvious (or natural) way and a slight variation of Definition 6.1.2 is more convenient in such a situation:

Definition 6.1.4 Given a set, M, and any two integers, $n \geq 1$ and $k \geq 1$, a C^k *n-atlas* (or *n-atlas of class* C^k), $\mathcal{A}$, is a family of charts, $\{(U_i, \varphi_i)\}$, such that

(1) Each U_i is a subset of M and $\varphi_i \colon U_i \to \varphi_i(U_i)$ is a bijection onto an open subset, $\varphi_i(U_i) \subseteq \mathbb{R}^n$, for all i;

(2) The U_i cover M, i.e.,

$$M = \bigcup_i U_i;$$

(3) Whenever $U_i \cap U_j \neq \emptyset$, the set $\varphi_i(U_i \cap U_j)$ is open in $\mathbb{R}^n$ and the transition map φ_i^j (and φ_j^i) is a C^k-diffeomorphism.

Then, the notion of a chart being compatible with an atlas and of two atlases being compatible is just as before and we get a new definition of a manifold, analogous to Definition 6.1.3.

But, this time, we give M the topology in which the open sets are arbitrary unions of domains of charts, U_i, more precisely, the U_i's of the maximal atlas defining the differentiable structure on M.

It is not difficult to verify that the axioms of a topology are verified and M is indeed a topological space with this topology.

It can also be shown that when M is equipped with the above topology, then the maps $\varphi_i \colon U_i \to \varphi_i(U_i)$ are homeomorphisms, so M is a manifold according to Definition 6.1.3. Thus, we are back to the original notion of a manifold where it is assumed that M is already a topological space.

One can also define the topology on M in terms of any the atlases, $\mathcal{A}$, defining M (not only the maximal one) by requiring $U \subseteq M$ to be open iff $\varphi_i(U \cap U_i)$ is open in $\mathbb{R}^n$, for every chart, (U_i, φ_i), in the altas $\mathcal{A}$. This topology is the same as the topology induced by the maximal atlas.

We also require M to be Hausdorff and second-countable with this topology. If M has a countable atlas, then it is second-countable

If the underlying topological space of a manifold is compact, then M has some finite atlas.

Also, if $\mathcal{A}$ is some atlas for M and (U, φ) is a chart in $\mathcal{A}$, for any (nonempty) open subset, $V \subseteq U$, we get a chart, $(V, \varphi \restriction V)$, and it is obvious that this chart is compatible with $\mathcal{A}$.

Thus, $(V, \varphi \restriction V)$ is also a chart for M. This observation shows that if U is any open subset of a C^k-manifold, M, then U is also a C^k-manifold whose charts are the restrictions of charts on M to U.

Example 1. The sphere S^n.

Using the stereographic projections (from the north pole and the south pole), we can define two charts on S^n and show that S^n is a smooth manifold. Let $\sigma_N \colon S^n - \{N\} \to \mathbb{R}^n$ and $\sigma_S \colon S^n - \{S\} \to \mathbb{R}^n$, where $N = (0, \cdots, 0, 1) \in \mathbb{R}^{n+1}$ (the north pole) and $S = (0, \cdots, 0, -1) \in \mathbb{R}^{n+1}$ (the south pole) be the maps called respectively *stereographic projection from the north pole* and *stereographic projection from the south pole* given by

$$\sigma_N(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}\,(x_1, \ldots, x_n)$$

and

$$\sigma_S(x_1, \ldots, x_{n+1}) = \frac{1}{1 + x_{n+1}}\,(x_1, \ldots, x_n).$$

The inverse stereographic projections are given by

$$\sigma_N^{-1}(x_1, \ldots, x_n) = \frac{1}{(\sum_{i=1}^n x_i^2) + 1}\,\Big(2x_1, \ldots, 2x_n, (\sum_{i=1}^n x_i^2) - 1\Big)$$

and

$$\sigma_S^{-1}(x_1, \ldots, x_n) = \frac{1}{(\sum_{i=1}^n x_i^2) + 1}\,\Big(2x_1, \ldots, 2x_n, -(\sum_{i=1}^n x_i^2) + 1\Big).$$

Thus, if we let $U_N = S^n - \{N\}$ and $U_S = S^n - \{S\}$, we see that U_N and U_S are two open subsets covering S^n, both homeomorphic to $\mathbb{R}^n$.

Furthermore, it is easily checked that on the overlap, $U_N \cap U_S = S^n - \{N, S\}$, the transition maps

$$\sigma_S \circ \sigma_N^{-1} = \sigma_N \circ \sigma_S^{-1}$$

are given by

$$(x_1, \ldots, x_n) \mapsto \frac{1}{\sum_{i=1}^n x_i^2}(x_1, \ldots, x_n),$$

that is, the inversion of center $O = (0, \ldots, 0)$ and power 1. Clearly, this map is smooth on $\mathbb{R}^n - \{O\}$, so we conclude that (U_N, σ_N) and (U_S, σ_S) form a smooth atlas for S^n.

Example 2. The projective space $\mathbb{RP}^n$.

To define an atlas on $\mathbb{RP}^n$ it is convenient to view $\mathbb{RP}^n$ as the set of equivalence classes of vectors in $\mathbb{R}^{n+1} - \{0\}$ modulo the equivalence relation,

$$u \sim v \quad \text{iff} \quad v = \lambda u, \quad \text{for some} \quad \lambda \neq 0 \in \mathbb{R}.$$

Given any $p = [x_1, \ldots, x_{n+1}] \in \mathbb{RP}^n$, we call $(x_1, \ldots, x_{n+1})$ the *homogeneous coordinates* of p.

It is customary to write $(x_1 \colon \cdots \colon x_{n+1})$ instead of $[x_1, \ldots, x_{n+1}]$. (Actually, in most books, the indexing starts with 0, i.e., homogeneous coordinates for $\mathbb{RP}^n$ are written as $(x_0 \colon \cdots \colon x_n)$.)

For any i, with $1 \leq i \leq n+1$, let

$$U_i = \{(x_1 \colon \cdots \colon x_{n+1}) \in \mathbb{RP}^n \mid x_i \neq 0\}.$$

Observe that U_i is well defined, because if $(y_1 \colon \cdots \colon y_{n+1}) = (x_1 \colon \cdots \colon x_{n+1})$, then there is some $\lambda \neq 0$ so that $y_i = \lambda x_i$, for $i = 1, \ldots, n+1$.

We can define a homeomorphism, φ_i, of U_i onto $\mathbb{R}^n$, as follows:

$$\varphi_i(x_1 \colon \cdots \colon x_{n+1}) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right),$$

where the ith component is omitted. Again, it is clear that this map is well defined since it only involves ratios.

We can also define the maps, ψ_i, from $\mathbb{R}^n$ to $U_i \subseteq \mathbb{RP}^n$, given by

$$\psi_i(x_1, \ldots, x_n) = (x_1 \colon \cdots \colon x_{i-1} \colon 1 \colon x_i \colon \cdots \colon x_n),$$

where the 1 goes in the ith slot, for $i = 1, \ldots, n+1$.

One easily checks that φ_i and ψ_i are mutual inverses, so the φ_i are homeomorphisms. On the overlap, $U_i \cap U_j$, (where $i \neq j$), as $x_j \neq 0$, we have

$$(\varphi_j \circ \varphi_i^{-1})(x_1, \ldots, x_n) = \left(\frac{x_1}{x_j}, \ldots, \frac{x_{i-1}}{x_j}, \frac{1}{x_j}, \frac{x_i}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j}\right).$$

(We assumed that $i < j$; the case $j < i$ is similar.) This is clearly a smooth function from $\varphi_i(U_i \cap U_j)$ to $\varphi_j(U_i \cap U_j)$.

As the U_i cover $\mathbb{RP}^n$, see conclude that the (U_i, φ_i) are $n+1$ charts making a smooth atlas for $\mathbb{RP}^n$.

Intuitively, the space $\mathbb{RP}^n$ is obtained by glueing the open subsets U_i on their overlaps. Even for $n = 3$, this is not easy to visualize!

Example 3. The Grassmannian $G(k, n)$.

Recall that $G(k, n)$ is the set of all k-dimensional linear subspaces of $\mathbb{R}^n$, also called k-planes.

Every k-plane, W, is the linear span of k linearly independent vectors, $u_1, \ldots, u_k$, in $\mathbb{R}^n$; furthermore, $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ both span W iff there is an invertible $k \times k$-matrix, $\Lambda = (\lambda_{ij})$, such that

$$v_i = \sum_{j=1}^{k} \lambda_{ij} u_j, \quad 1 \leq i \leq k.$$

Obviously, there is a bijection between the collection of k linearly independent vectors, $u_1, \ldots, u_k$, in $\mathbb{R}^n$ and the collection of $n \times k$ matrices of rank k.

Furthermore, two $n \times k$ matrices A and B of rank k represent the same k-plane iff

$$B = A\Lambda, \quad \text{for some invertible } k \times k \text{ matrix, } \Lambda.$$

(Note the analogy with projective spaces where two vectors u, v represent the same point iff $v = \lambda u$ for some invertible $\lambda \in \mathbb{R}$.)

We can define the domain of charts (according to Definition 6.1.4) on $G(k, n)$ as follows: For every subset, $S = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$, let U_S be the subset of $n \times k$ matrices, A, of rank k whose rows of index in $S = \{i_1, \ldots, i_k\}$ forms an invertible $k \times k$ matrix denoted A_S.

Observe that the $k \times k$ matrix consisting of the rows of the matrix AA_S^{-1} whose index belong to S is the identity matrix, I_k.

Therefore, we can define a map, $\varphi_S \colon U_S \to \mathbb{R}^{(n-k)\times k}$, where $\varphi_S(A) =$ the $(n-k)\times k$ matrix obtained by deleting the rows of index in S from AA_S^{-1}.

We need to check that this map is well defined, i.e., that it does not depend on the matrix, A, representing W.

Let us do this in the case where $S = \{1, \ldots, k\}$, which is notationally simpler. The general case can be reduced to this one using a suitable permutation.

If $B = A\Lambda$, with Λ invertible, if we write

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

as $B = A\Lambda$, we get $B_1 = A_1\Lambda$ and $B_2 = A_2\Lambda$, from which we deduce that

$$\begin{aligned} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} B_1^{-1} = \begin{pmatrix} I_k \\ B_2 B_1^{-1} \end{pmatrix} = \\ \begin{pmatrix} I_k \\ A_2 \Lambda\Lambda^{-1} A_1^{-1} \end{pmatrix} = \begin{pmatrix} I_k \\ A_2 A_1^{-1} \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} A_1^{-1}. \end{aligned}$$

Therefore, our map is indeed well-defined.

It is clearly injective and we can define its inverse, ψ_S, as follows: Let π_S be the permutation of $\{1, \ldots, n\}$ swaping $\{1, \ldots, k\}$ and S and leaving every other element fixed (i.e., if $S = \{i_1, \ldots, i_k\}$, then $\pi_S(j) = i_j$ and $\pi_S(i_j) = j$, for $j = 1, \ldots, k$).

If P_S is the permutation matrix associated with π_S, for any $(n-k) \times k$ matrix, M, let

$$\psi_S(M) = P_S \begin{pmatrix} I_k \\ M \end{pmatrix}.$$

The effect of ψ_S is to "insert into M" the rows of the identity matrix I_k as the rows of index from S.

At this stage, we have charts that are bijections from subsets, U_S, of $G(k, n)$ to open subsets, namely, $\mathbb{R}^{(n-k)\times k}$.

Then, the reader can check that the transition map $\varphi_T \circ \varphi_S^{-1}$ from $\varphi_S(U_S \cap U_U)$ to $\varphi_T(U_S \cap U_U)$ is given by

$$M \mapsto (C + DM)(A + BM)^{-1},$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = P_T P_S,$$

is the matrix of the permutation $\pi_T \circ \pi_S$ (this permutation "shuffles" S and T).

This map is smooth, as it is given by determinants, and so, the charts (U_S, φ_S) form a smooth atlas for $G(k, n)$.

Finally, one can check that the conditions of Definition 6.1.4 are satisfied, so the atlas just defined makes $G(k, n)$ into a topological space and a smooth manifold.

Remark: The reader should have no difficulty proving that the collection of k-planes represented by matrices in U_S is precisely set of k-planes, W, supplementary to the $(n-k)$-plane spanned by the $n-k$ canonical basis vectors $e_{j_{k+1}}, \ldots, e_{j_n}$ (i.e., $\mathrm{span}(W \cup \{e_{j_{k+1}}, \ldots, e_{j_n}\}) = \mathbb{R}^n$, where $S = \{i_1, \ldots, i_k\}$ and $\{j_{k+1}, \ldots, j_n\} = \{1, \ldots, n\} - S$).

Example 4. Product Manifolds.

Let M_1 and M_2 be two C^k-manifolds of dimension n_1 and n_2, respectively.

The topological space, $M_1 \times M_2$, with the product topology (the opens of $M_1 \times M_2$ are arbitrary unions of sets of the form $U \times V$, where U is open in M_1 and V is open in M_2) can be given the structure of a C^k-manifold of dimension $n_1 + n_2$ by defining charts as follows:

For any two charts, (U_i, φ_i) on M_1 and (V_j, ψ_j) on M_2, we declare that $(U_i \times V_j, \varphi_i \times \psi_j)$ is a chart on $M_1 \times M_2$, where $\varphi_i \times \psi_j \colon U_i \times V_j \to \mathbb{R}^{n_1+n_2}$ is defined so that

$$\varphi_i \times \psi_j(p, q) = (\varphi_i(p), \psi_j(q)), \quad \text{for all } (p, q) \in U_i \times V_j.$$

We define C^k-maps between manifolds as follows:

Definition 6.1.5 Given any two C^k-manifolds, M and N, of dimension m and n respectively, a *C^k-map* if a continuous functions, $h: M \to N$, so that for every $p \in M$, there is some chart, (U, φ), at p and some chart, (V, ψ), at $q = h(p)$, with $f(U) \subseteq V$ and

$$\psi \circ h \circ \varphi^{-1}: \varphi(U) \longrightarrow \psi(V)$$

a C^k-function.

It is easily shown that Definition 6.1.5 does not depend on the choice of charts. In particular, if $N = \mathbb{R}$, we obtain a *C^k-function on M*.

One checks immediately that a function, $f: M \to \mathbb{R}$, is a C^k-map iff for every $p \in M$, there is some chart, (U, φ), at p so that

$$f \circ \varphi^{-1}: \varphi(U) \longrightarrow \mathbb{R}$$

is a C^k-function.

If U is an open subset of M, set of C^k-functions on U is denoted by $\mathcal{C}^k(U)$. In particular, $\mathcal{C}^k(M)$ denotes the set of C^k-functions on the manifold, M. Observe that $\mathcal{C}^k(U)$ is a ring.

On the other hand, if M is an open interval of $\mathbb{R}$, say $M =]a, b[$, then $\gamma\colon]a, b[\to N$ is called a C^k-*curve* in N. One checks immediately that a function, $\gamma\colon]a, b[\to N$, is a C^k-map iff for every $q \in N$, there is some chart, (V, ψ), at q so that

$$\psi \circ \gamma\colon]a, b[\longrightarrow \psi(V)$$

is a C^k-function.

It is clear that the composition of C^k-maps is a C^k-map. A C^k-map, $h\colon M \to N$, between two manifolds is a C^k-*diffeomorphism* iff h has an inverse, $h^{-1}\colon N \to M$ (i.e., $h^{-1} \circ h = \mathrm{id}_M$ and $h \circ h^{-1} = \mathrm{id}_N$), and both h and h^{-1} are C^k-maps (in particular, h and h^{-1} are homeomorphisms). Next, we define tangent vectors.

6.2 Tangent Vectors, Tangent Spaces, Cotangent Spaces

Let M be a C^k manifold of dimension n, with $k \geq 1$. The most intuitive method to define tangent vectors is to use curves. Let $p \in M$ be any point on M and let $\gamma\colon]-\epsilon, \epsilon[\to M$ be a C^1-curve passing through p, that is, with $\gamma(0) = p$. Unfortunately, if M is not embedded in any $\mathbb{R}^N$, the derivative $\gamma'(0)$ does not make sense. However, for any chart, (U, φ), at p, the map $\varphi \circ \gamma$ is a C^1-curve in $\mathbb{R}^n$ and the tangent vector $v = (\varphi \circ \gamma)'(0)$ is well defined. The trouble is that different curves may yield the same v!

To remedy this problem, we define an equivalence relation on curves through p as follows:

Definition 6.2.1 Given a C^k manifold, M, of dimension n, for any $p \in M$, two C^1-curves, $\gamma_1\colon]-\epsilon_1, \epsilon_1[\to M$ and $\gamma_2\colon]-\epsilon_2, \epsilon_2[\to M$, through p (i.e., $\gamma_1(0) = \gamma_2(0) = p$) are *equivalent* iff there is some chart, (U, φ), at p so that

$$(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

Now, the problem is that this definition seems to depend on the choice of the chart. Fortunately, this is not the case.

This leads us to the first definition of a tangent vector.

Definition 6.2.2 *(Tangent Vectors, Version 1)* Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, a *tangent vector to M at p* is any equivalence class of C^1-curves through p on M, modulo the equivalence relation defined in Definition 6.2.1. The set of all tangent vectors at p is denoted by $T_p(M)$.

It is obvious that $T_p(M)$ is a vector space.

We will show that $T_p(M)$ is a vector space of dimension $n =$ dimension of M.

One should observe that unless $M = \mathbb{R}^n$, in which case, for any $p, q \in \mathbb{R}^n$, the tangent space $T_q(M)$ is naturally isomorphic to the tangent space $T_p(M)$ by the translation $q - p$, for an arbitrary manifold, there is no relationship between $T_p(M)$ and $T_q(M)$ when $p \neq q$.

One of the defects of the above definition of a tangent vector is that it has no clear relation to the C^k-differential structure of M.

In particular, the definition does not seem to have anything to do with the functions defined locally at p.

There is another way to define tangent vectors that reveals this connection more clearly. Moreover, such a definition is more intrinsic, i.e., does not refer explicitly to charts.

As a first step, consider the following: Let (U, φ) be a chart at $p \in M$ (where M is a C^k-manifold of dimension n, with $k \geq 1$) and let $x_i = pr_i \circ \varphi$, the ith local coordinate ($1 \leq i \leq n$).

For any function, f, defined on $U \ni p$, set

$$\left(\frac{\partial}{\partial x_i}\right)_p f = \left.\frac{\partial (f \circ \varphi^{-1})}{\partial X_i}\right|_{\varphi(p)}, \qquad 1 \leq i \leq n.$$

(Here, $(\partial g/\partial X_i)|_y$ denotes the partial derivative of a function $g\colon \mathbb{R}^n \to \mathbb{R}$ with respect to the ith coordinate, evaluated at y.)

We would expect that the function that maps f to the above value is a linear map on the set of functions defined locally at p, but there is technical difficulty:

The set of functions defined locally at p is **not** a vector space!

To see this, observe that if f is defined on an open $U \ni p$ and g is defined on a different open $V \ni p$, then we do know how to define $f + g$.

The problem is that we need to identify functions that agree on a smaller open. This leads to the notion of *germs*.

Definition 6.2.3 Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, a *locally defined function at* p is a pair, (U, f), where U is an open subset of M containing p and f is a function defined on U. Two locally defined functions, (U, f) and (V, g), at p are *equivalent* iff there is some open subset, $W \subseteq U \cap V$, containing p so that

$$f \upharpoonright W = g \upharpoonright W.$$

The equivalence class of a locally defined function at p, denoted $[f]$ or $\mathbf{f}$, is called a *germ at* p.

One should check that the relation of Definition 6.2.3 is indeed an equivalence relation.

Of course, the value at p of all the functions, f, in any germ, $\mathbf{f}$, is $f(p)$. Thus, we set $\mathbf{f}(p) = f(p)$.

One should also check that we can define addition of germs, multiplication of a germ by a scalar and multiplication of germs, in the obvious way:

If $\mathbf{f}$ and $\mathbf{g}$ are two germs at p, and $\lambda \in \mathbb{R}$, then

$$\begin{aligned} [f] + [g] &= [f + g] \\ \lambda[f] &= [\lambda f] \\ [f][g] &= [fg]. \end{aligned}$$

(Of course, $f + g$ is the function locally defined so that $(f+g)(x) = f(x)+g(x)$ and similarly, $(\lambda f)(x) = \lambda f(x)$ and $(fg)(x) = f(x)g(x)$.)

Therefore, the germs at p form a ring. The ring of germs of C^k-functions at p is denoted $\mathcal{O}^{(k)}_{M,p}$. When $k = \infty$, we usually drop the superscript ∞.

Remark: Most readers will most likely be puzzled by the notation $\mathcal{O}^{(k)}_{M,p}$.

In fact, it is standard in algebraic geometry, but it is not as commonly used in differential geometry.

For any open subset, U, of a manifold, M, the ring, $\mathcal{C}^k(U)$, of C^k-functions on U is also denoted $\mathcal{O}_M^{(k)}(U)$ (certainly by people with an algebraic geometry bent!).

Then, it turns out that the map $U \mapsto \mathcal{O}_M^{(k)}(U)$ is a *sheaf*, denoted $\mathcal{O}_M^{(k)}$, and the ring $\mathcal{O}_{M,p}^{(k)}$ is the *stalk* of the sheaf $\mathcal{O}_M^{(k)}$ at p.

Such rings are called *local rings*. Roughly speaking, all the "local" information about M at p is contained in the local ring $\mathcal{O}_{M,p}^{(k)}$. (This is to be taken with a grain of salt. In the C^k-case where $k < \infty$, we also need the "stationary germs", as we will see shortly.)

Now that we have a rigorous way of dealing with functions locally defined at p, observe that the map

$$v_i \colon f \mapsto \left(\frac{\partial}{\partial x_i}\right)_p f$$

yields the same value for all functions f in a germ $\mathbf{f}$ at p.

Furthermore, the above map is linear on $\mathcal{O}_{M,p}^{(k)}$. More is true.

Firstly for any two functions f, g locally defined at p, we have

$$\left(\frac{\partial}{\partial x_i}\right)_p (fg) = f(p)\left(\frac{\partial}{\partial x_i}\right)_p g + g(p)\left(\frac{\partial}{\partial x_i}\right)_p f.$$

Secondly, if $(f \circ \varphi^{-1})'(\varphi(p)) = 0$, then

$$\left(\frac{\partial}{\partial x_i}\right)_p f = 0.$$

The first property says that v_i is a *derivation*. As to the second property, when $(f \circ \varphi^{-1})'(\varphi(p)) = 0$, we say that *$f$ is stationary at p*.

It is easy to check (using the chain rule) that being stationary at p does not depend on the chart, (U, φ), at p or on the function chosen in a germ, $\mathbf{f}$. Therefore, the notion of a stationary germ makes sense:

We say that $\mathbf{f}$ *is a stationary germ* iff $(f \circ \varphi^{-1})'(\varphi(p)) = 0$ for some chart, (U, φ), at p and some function, f, in the germ, $\mathbf{f}$.

The C^k-stationary germs form a subring of $\mathcal{O}_{M,p}^{(k)}$ (but not an ideal!) denoted $\mathcal{S}_{M,p}^{(k)}$.

Remarkably, it turns out that the dual of the vector space, $\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)}$, is isomorphic to the tangent space, $T_p(M)$.

First, we prove that the subspace of linear forms on $\mathcal{O}_{M,p}^{(k)}$ that vanish on $\mathcal{S}_{M,p}^{(k)}$ has $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$ as a basis.

Proposition 6.2.4 *Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$ and any chart (U, φ) at p, the n functions, $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$, defined on $\mathcal{O}_{M,p}^{(k)}$ by*

$$\left(\frac{\partial}{\partial x_i}\right)_p f = \left.\frac{\partial(f \circ \varphi^{-1})}{\partial X_i}\right|_{\varphi(p)}, \qquad 1 \leq i \leq n$$

are linear forms that vanish on $\mathcal{S}_{M,p}^{(k)}$. Every linear form, L, on $\mathcal{O}_{M,p}^{(k)}$ that vanishes on $\mathcal{S}_{M,p}^{(k)}$ can be expressed in a unique way as

$$L = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p,$$

where $\lambda_i \in \mathbb{R}$. Therefore, the

$$\left(\frac{\partial}{\partial x_i}\right)_p, \qquad i = 1, \ldots, n$$

form a basis of the vector space of linear forms on $\mathcal{O}_{M,p}^{(k)}$ that vanish on $\mathcal{S}_{M,p}^{(k)}$.

As the subspace of linear forms on $\mathcal{O}^{(k)}_{M,p}$ that vanish on $\mathcal{S}^{(k)}_{M,p}$ is isomorphic to the dual, $(\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})^*$, of the space $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$, we see that the

$$\left(\frac{\partial}{\partial x_i}\right)_p, \qquad i = 1, \ldots, n$$

also form a basis of $(\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})^*$.

To define our second version of tangent vectors, we need to define linear derivations.

Definition 6.2.5 Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, a *linear derivation at p* is a linear form, v, on $\mathcal{O}^{(k)}_{M,p}$, such that

$$v(\mathbf{fg}) = f(p)v(\mathbf{g}) + g(p)v(\mathbf{f}),$$

for all germs $\mathbf{f}, \mathbf{g} \in \mathcal{O}^{(k)}_{M,p}$. The above is called the *Leibnitz property*.

Recall that we observed earlier that the $\left(\frac{\partial}{\partial x_i}\right)_p$ are linear derivations at p. Therefore, we have

Proposition 6.2.6 *Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, the linear forms on $\mathcal{O}_{M,p}^{(k)}$ that vanish on $\mathcal{S}_{M,p}^{(k)}$ are exactly the linear derivations on $\mathcal{O}_{M,p}^{(k)}$ that vanish on $\mathcal{S}_{M,p}^{(k)}$.*

Here is now our second definition of a tangent vector.

Definition 6.2.7 *(Tangent Vectors, Version 2)* Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, a *tangent vector to M at p* is any linear derivation on $\mathcal{O}_{M,p}^{(k)}$ that vanishes on $\mathcal{S}_{M,p}^{(k)}$, the subspace of stationary germs.

Let us consider the simple case where $M = \mathbb{R}$. In this case, for every $x \in \mathbb{R}$, the tangent space, $T_x(\mathbb{R})$, is a one-dimensional vector space isomorphic to $\mathbb{R}$ and $\left(\frac{\partial}{\partial t}\right)_x = \frac{d}{dt}\Big|_x$ is a basis vector of $T_x(\mathbb{R})$.

For every C^k-function, f, locally defined at x, we have

$$\left(\frac{\partial}{\partial t}\right)_x f = \left.\frac{df}{dt}\right|_x = f'(x).$$

Thus, $\left(\frac{\partial}{\partial t}\right)_x$ is: compute the derivative of a function at x.

We now prove the equivalence of the two Definitions of a tangent vector.

Proposition 6.2.8 *Let M be any C^k-manifold of dimension n, with $k \geq 1$. For any $p \in M$, let u be any tangent vector (version 1) given by some equivalence class of C^1-curves, $\gamma\colon]-\epsilon, +\epsilon[\to M$, through p (i.e., $p = \gamma(0)$). Then, the map L_u defined on $\mathcal{O}_{M,p}^{(k)}$ by*

$$L_u(\mathbf{f}) = (f \circ \gamma)'(0)$$

is a linear derivation that vanishes on $\mathcal{S}_{M,p}^{(k)}$. Furthermore, the map $u \mapsto L_u$ defined above is an isomorphism between $T_p(M)$ and $(\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)})^*$, the space of linear forms on $\mathcal{O}_{M,p}^{(k)}$ that vanish on $\mathcal{S}_{M,p}^{(k)}$.

In view of Proposition 6.2.8, we can identify $T_p(M)$ with $(\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})^*$.

As the space $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$ is finite dimensional, $(\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})^{**}$ is canonically isomorphic to $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$, so we can identify $T^*_p(M)$ with $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$.

(Recall that if E is a finite dimensional space, the map $i_E\colon E \to E^{**}$ defined so that, for any $v \in E$,

$$v \mapsto \widetilde{v}, \quad \text{where} \quad \widetilde{v}(f) = f(v), \quad \text{for all } f \in E^*$$

is a linear isomorphism.) This also suggests the following definition:

Definition 6.2.9 Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, the *tangent space at p*, denoted $T_p(M)$, is the space of linear derivations on $\mathcal{O}^{(k)}_{M,p}$ that vanish on $\mathcal{S}^{(k)}_{M,p}$. Thus, $T_p(M)$ can be identified with $(\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})^*$. The space $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$ is called the *cotangent space at p*; it is isomorphic to the dual, $T^*_p(M)$, of $T_p(M)$.

Observe that if $x_i = pr_i \circ \varphi$, as

$$\left(\frac{\partial}{\partial x_i}\right)_p x_j = \delta_{i,j},$$

the images of $x_1, \ldots, x_n$ in $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$ are the dual of the basis $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$ of $T_p(M)$.

Given any C^k-function, f, on M, we denote the image of f in $T_p^*(M) = \mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$ by df_p.

This is the *differential of f at p.*

Using the isomorphism between $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$ and $(\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})^{**}$ described above, df_p corresponds to the linear map in $T_p^*(M)$ defined by $df_p(v) = v(\mathbf{f})$, for all $v \in T_p(M)$.

With this notation, we see that $(dx_1)_p, \ldots, (dx_n)_p$ is a basis of $T_p^*(M)$, and this basis is dual to the basis $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$ of $T_p(M)$.

For simplicity of notation, we often omit the subscript p unless confusion arises.

Remark: Strictly speaking, a tangent vector, $v \in T_p(M)$, is defined on the space of germs, $\mathcal{O}^{(k)}_{M,p}$ at p. However, it is often convenient to define v on C^k-functions $f \in \mathcal{C}^k(U)$, where U is some open subset containing p. This is easy: Set

$$v(f) = v(\mathbf{f}).$$

Given any chart, (U, φ), at p, since v can be written in a unique way as

$$v = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p,$$

we get

$$v(f) = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p f.$$

This shows that $v(f)$ is the *directional derivative of* f *in the direction* v.

When M is a smooth manifold, things get a little simpler. Indeed, it turns out that in this case, every linear derivation vanishes on stationary germs.

To prove this, we recall the following result from calculus (see Warner [**?**]):

Proposition 6.2.10 *If $g\colon \mathbb{R}^n \to \mathbb{R}$ is a C^k-function ($k \geq 2$) on a convex open, U, about $p \in \mathbb{R}^n$, then for every $q \in U$, we have*

$$g(q) = g(p) + \sum_{i=1}^{n} \left.\frac{\partial g}{\partial X_i}\right|_p (q_i - p_i)$$
$$+ \sum_{i,j=1}^{n} (q_i - p_i)(q_j - p_j) \int_0^1 (1-t) \left.\frac{\partial^2 g}{\partial X_i \partial X_j}\right|_{(1-t)p+tq} dt.$$

In particular, if $g \in C^\infty(U)$, then the integral as a function of q is C^∞.

Proposition 6.2.11 *Let M be any C^∞-manifold of dimension n. For any $p \in M$, any linear derivation on $\mathcal{O}_{M,p}^{(\infty)}$ vanishes on stationary germs.*

Proposition 6.2.11 shows that in the case of a smooth manifold, in Definition 6.2.7, we can omit the requirement that linear derivations vanish on stationary germs, since this is automatic.

It is also possible to define $T_p(M)$ just in terms of $\mathcal{O}^{(\infty)}_{M,p}$.

Let $\mathfrak{m}_{M,p} \subseteq \mathcal{O}^{(\infty)}_{M,p}$ be the ideal of germs that vanish at p. Then, we also have the ideal $\mathfrak{m}^2_{M,p}$, which consists of all finite sums of products of two elements in $\mathfrak{m}_{M,p}$, and it can be shown that $T^*_p(M)$ is isomorphic to $\mathfrak{m}_{M,p}/\mathfrak{m}^2_{M,p}$ (see Warner [**?**], Lemma 1.16).

Actually, if we let $\mathfrak{m}^{(k)}_{M,p}$ denote the C^k germs that vanish at p and $\mathfrak{s}^{(k)}_{M,p}$ denote the stationary C^k-germs that vanish at p, it is easy to show that

$$\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p} \cong \mathfrak{m}^{(k)}_{M,p}/\mathfrak{s}^{(k)}_{M,p}.$$

(Given any $\mathbf{f} \in \mathcal{O}^{(k)}_{M,p}$, send it to $\mathbf{f} - \mathbf{f}(\mathbf{p}) \in \mathfrak{m}^{(k)}_{M,p}$.)

Clearly, $(\mathfrak{m}_{M,p}^{(k)})^2$ consists of stationary germs (by the derivation property) and when $k = \infty$, Proposition 6.2.10 shows that every stationary germ that vanishes at p belongs to $\mathfrak{m}_{M,p}^2$. Therefore, when $k = \infty$, we have $\mathfrak{s}_{M,p}^{(\infty)} = \mathfrak{m}_{M,p}^2$ and so,

$$T_p^*(M) = \mathcal{O}_{M,p}^{(\infty)}/\mathcal{S}_{M,p}^{(\infty)} \cong \mathfrak{m}_{M,p}/\mathfrak{m}_{M,p}^2.$$

Remark: The ideal $\mathfrak{m}_{M,p}^{(k)}$ is in fact the unique maximal ideal of $\mathcal{O}_{M,p}^{(k)}$.

Thus, $\mathcal{O}_{M,p}^{(k)}$ is a local ring (in the sense of commutative algebra) called the *local ring of germs of C^k-functions at p*. These rings play a crucial role in algebraic geometry.

Yet one more way of defining tangent vectors will make it a little easier to define tangent bundles.

Definition 6.2.12 *(Tangent Vectors, Version 3)* Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, consider the triples, (U, φ, u), where (U, φ) is any chart at p and u is any vector in $\mathbb{R}^n$. Say that two such triples (U, φ, u) and (V, ψ, v) are *equivalent* iff

$$(\psi \circ \varphi^{-1})'_{\varphi(p)}(u) = v.$$

A *tangent vector* to M at p is an equivalence class of triples, $[(U, \varphi, u)]$, for the above equivalence relation.

The intuition behind Definition 6.2.12 is quite clear: The vector u is considered as a tangent vector to $\mathbb{R}^n$ at $\varphi(p)$.

If (U, φ) is a chart on M at p, we can define a natural isomorphism, $\theta_{U,\varphi,p} \colon \mathbb{R}^n \to T_p(M)$, between $\mathbb{R}^n$ and $T_p(M)$, as follows: For any $u \in \mathbb{R}^n$,

$$\theta_{U,\varphi,p} \colon u \mapsto [(U, \varphi, u)].$$

One immediately check that the above map is indeed linear and a bijection.

The equivalence of this definition with the definition in terms of curves (Definition 6.2.2) is easy to prove.

Proposition 6.2.13 *Let M be any C^k-manifold of dimension n, with $k \geq 1$. For any $p \in M$, let x be any tangent vector (version 1) given by some equivalence class of C^1-curves, $\gamma\colon]-\epsilon, +\epsilon[\to M$, through p (i.e., $p = \gamma(0)$). The map*

$$x \mapsto [(U, \varphi, (\varphi \circ \gamma)'(0))]$$

is an isomorphism between $T_p(M)$-version 1 and $T_p(M)$-version 3.

For simplicity of notation, we also use the notation T_pM for $T_p(M)$ (resp. T_p^*M for $T_p^*(M)$).

After having explored thorougly the notion of tangent vector, we show how a C^k-map, $h\colon M \to N$, between C^k manifolds, induces a linear map, $dh_p\colon T_p(M) \to T_{h(p)}(N)$, for every $p \in M$.

We find it convenient to use Version 2 of the definition of a tangent vector. So, let $u \in T_p(M)$ be a linear derivation on $\mathcal{O}^{(k)}_{M,p}$ that vanishes on $\mathcal{S}^{(k)}_{M,p}$.

We would like $dh_p(u)$ to be a linear derivation on $\mathcal{O}^{(k)}_{N,h(p)}$ that vanishes on $\mathcal{S}^{(k)}_{N,h(p)}$.

So, for every germ, $\mathbf{g} \in \mathcal{O}^{(k)}_{N,h(p)}$, set

$$dh_p(u)(\mathbf{g}) = u(\mathbf{g} \circ \mathbf{h}).$$

For any locally defined function, g, at $h(p)$ in the germ, $\mathbf{g}$ (at $h(p)$), it is clear that $g \circ h$ is locally defined at p and is C^k, so $\mathbf{g} \circ \mathbf{h}$ is indeed a C^k-germ at p.

Moreover, if $\mathbf{g}$ is a stationary germ at $h(p)$, then for some chart, (V, ψ) on N at $q = h(p)$, we have $(g \circ \psi^{-1})'(\psi(q)) = 0$ and, for some chart (U, φ) at p on M, we get

$$(g \circ h \circ \varphi^{-1})'(\varphi(p)) = (g \circ \psi^{-1})(\psi(q))((\psi \circ h \circ \varphi^{-1})'(\varphi(p))) \\ = 0,$$

which means that $\mathbf{g} \circ \mathbf{h}$ is stationary at p.

Therefore, $dh_p(u) \in T_{h(p)}(M)$. It is also clear that dh_p is a linear map. We summarize all this in the following definition:

Definition 6.2.14 Given any two C^k-manifolds, M and N, of dimension m and n, respectively, for any C^k-map, $h\colon M \to N$, and for every $p \in M$, the *differential of h at p* or *tangent map*, $dh_p\colon T_p(M) \to T_{h(p)}(N)$, is the linear map defined so that

$$dh_p(u)(\mathbf{g}) = u(\mathbf{g} \circ \mathbf{h}),$$

for every $u \in T_p(M)$ and every germ, $\mathbf{g} \in \mathcal{O}^{(k)}_{N,h(p)}$. The linear map dh_p is also denoted $T_p h$ (and sometimes, h'_p or $D_p h$).

The chain rule is easily generalized to manifolds.

Proposition 6.2.15 *Given any two C^k-maps $f\colon M \to N$ and $g\colon N \to P$ between smooth C^k-manifolds, for any $p \in M$, we have*

$$d(g \circ f)_p = dg_{f(p)} \circ df_p.$$

In the special case where $N = \mathbb{R}$, a C^k-map between the manifolds M and $\mathbb{R}$ is just a C^k-function on M.

It is interesting to see what df_p is explicitly. Since $N = \mathbb{R}$, germs (of functions on $\mathbb{R}$) at $t_0 = f(p)$ are just germs of C^k-functions, $g\colon \mathbb{R} \to \mathbb{R}$, locally defined at t_0.

Then, for any $u \in T_p(M)$ and every germ $\mathbf{g}$ at t_0,

$$df_p(u)(\mathbf{g}) = u(\mathbf{g} \circ \mathbf{f}).$$

If we pick a chart, (U, φ), on M at p, we know that the $\left(\frac{\partial}{\partial x_i}\right)_p$ form a basis of $T_p(M)$, with $1 \leq i \leq n$.

Therefore, it is enough to figure out what $df_p(u)(\mathbf{g})$ is when $u = \left(\frac{\partial}{\partial x_i}\right)_p$. In this case,

$$df_p\left(\left(\frac{\partial}{\partial x_i}\right)_p\right)(\mathbf{g}) = \left.\frac{\partial(g \circ f \circ \varphi^{-1})}{\partial X_i}\right|_{\varphi(p)}.$$

Using the chain rule, we find that

$$df_p\left(\left(\frac{\partial}{\partial x_i}\right)_p\right)(\mathbf{g}) = \left(\frac{\partial}{\partial x_i}\right)_p f \left.\frac{dg}{dt}\right|_{t_0}.$$

Therefore, we have

$$df_p(u) = u(\mathbf{f}) \left.\frac{d}{dt}\right|_{t_0}.$$

This shows that we can identify df_p with the linear form in $T_p^*(M)$ defined by

$$df_p(v) = v(\mathbf{f}).$$

This is consistent with our previous definition of df_p as the image of f in $T_p^*(M) = \mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)}$ (as $T_p(M)$ is isomorphic to $(\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)})^*$).

In preparation for the definition of the flow of a vector field (which will be needed to define the exponential map in Lie group theory), we need to define the tangent vector to a curve on a manifold.

Given a C^k-curve, $\gamma\colon]a, b[\to M$, on a C^k-manifold, M, for any $t_0 \in]a, b[$, we would like to define the tangent vector to the curve γ at t_0 as a tangent vector to M at $p = \gamma(t_0)$.

We do this as follows: Recall that $\left.\frac{d}{dt}\right|_{t_0}$ is a basis vector of $T_{t_0}(\mathbb{R}) = \mathbb{R}$.

So, define the *tangent vector to the curve* γ *at* t, denoted $\dot{\gamma}(t_0)$ (or $\gamma'(t)$, or $\frac{d\gamma}{dt}(t_0)$) by

$$\dot{\gamma}(t) = d\gamma_t \left(\left.\frac{d}{dt}\right|_{t_0} \right).$$

Sometime, it is necessary to define curves (in a manifold) whose domain is not an open interval.

A map, $\gamma\colon [a, b] \to M$, is a C^k*-curve in* M if it is the restriction of some C^k-curve, $\widetilde{\gamma}\colon]a-\epsilon, b+\epsilon[\to M$, for some (small) $\epsilon > 0$.

Note that for such a curve (if $k \geq 1$) the tangent vector, $\dot{\gamma}(t)$, is defined for all $t \in [a, b]$,

A curve, $\gamma\colon [a, b] \to M$, is *piecewise* C^k iff there a sequence, $a_0 = a, a_1, \ldots, a_m = b$, so that the restriction of γ to each $[a_i, a_{i+1}]$ is a C^k-curve, for $i = 0, \ldots, m-1$.

6.3 Tangent and Cotangent Bundles, Vector Fields

Let M be a C^k-manifold (with $k \geq 2$). Roughly speaking, a vector field on M is the assignment, $p \mapsto \xi(p)$, of a tangent vector, $\xi(p) \in T_p(M)$, to a point $p \in M$.

Generally, we would like such assignments to have some smoothness properties when p varies in M, for example, to be C^l, for some l related to k.

Now, if the collection, $T(M)$, of all tangent spaces, $T_p(M)$, was a C^l-manifold, then it would be very easy to define what we mean by a C^l-vector field: We would simply require the maps, $\xi\colon M \to T(M)$, to be C^l.

If M is a C^k-manifold of dimension n, then we can indeed define make $T(M)$ into a C^{k-1}-manifold of dimension $2n$ and we now sketch this construction.

We find it most convenient to use Version 3 of the definition of tangent vectors, i.e., as equivalence classes of triple (U, φ, u).

First, we let $T(M)$ be the disjoint union of the tangent spaces $T_p(M)$, for all $p \in M$. There is a *natural projection*,

$$\pi : T(M) \to M, \quad \text{where} \quad \pi(v) = p \quad \text{if} \quad v \in T_p(M).$$

We still have to give $T(M)$ a topology and to define a C^{k-1}-atlas.

For every chart, (U, φ), of M (with U open in M) we define the function $\widetilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\widetilde{\varphi}(v) = (\varphi \circ \pi(v), \theta^{-1}_{U,\varphi,\pi(v)}(v)),$$

where $v \in \pi^{-1}(U)$ and $\theta_{U,\varphi,p}$ is the isomorphism between $\mathbb{R}^n$ and $T_p(M)$ described just after Definition 6.2.12.

It is obvious that $\widetilde{\varphi}$ is a bijection between $\pi^{-1}(U)$ and $\varphi(U) \times \mathbb{R}^n$, an open subset of $\mathbb{R}^{2n}$.

We give $T(M)$ the weakest topology that makes all the $\widetilde{\varphi}$ continuous, i.e., we take the collection of subsets of the form $\widetilde{\varphi}^{-1}(W)$, where W is any open subset of $\varphi(U)\times\mathbb{R}^n$, as a basis of the topology of $T(M)$.

One easily checks that $T(M)$ is Hausdorff and second-countable in this topology. If (U, φ) and (V, ψ) are overlapping charts, then the transition function

$$\widetilde{\psi}\circ\widetilde{\varphi}^{-1}\colon \varphi(U\cap V)\times\mathbb{R}^n \longrightarrow \psi(U\cap V)\times\mathbb{R}^n$$

is given by

$$\widetilde{\psi}\circ\widetilde{\varphi}^{-1}(p, u) = (\psi\circ\varphi^{-1}(p), (\psi\circ\varphi^{-1})'(u)).$$

It is clear that $\widetilde{\psi}\circ\widetilde{\varphi}^{-1}$ is a C^{k-1}-map. Therefore, $T(M)$ is indeed a C^{k-1}-manifold of dimension $2n$, called the *tangent bundle*.

Remark: Even if the manifold M is naturally embedded in $\mathbb{R}^N$ (for some $N \geq n = \dim(M)$), it is not at all obvious how to view the tangent bundle, $T(M)$, as embedded in $\mathbb{R}^{N'}$, for sone suitable N'. Hence, we see that the definition of an abtract manifold is unavoidable.

A similar construction can be carried out for the cotangent bundle.

In this case, we let $T^*(M)$ be the disjoint union of the cotangent spaces $T_p^*(M)$.

We also have a natural projection, $\pi\colon T^*(M) \to M$, and we can define charts as follows: For any chart, (U, φ), on M, we define the function $\widetilde{\varphi}\colon \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\widetilde{\varphi}(\tau) = \left(\varphi \circ \pi(\tau), \tau\left(\left(\frac{\partial}{\partial x_1}\right)_{\pi(\tau)}\right), \ldots, \tau\left(\left(\frac{\partial}{\partial x_n}\right)_{\pi(\tau)}\right)\right),$$

where $\tau \in \pi^{-1}(U)$ and the $\left(\frac{\partial}{\partial x_i}\right)_p$ are the basis of $T_p(M)$ associated with the chart (U, φ).

Again, one can make $T^*(M)$ into a C^{k-1}-manifold of dimension $2n$, called the *cotangent bundle*.

Observe that for every chart, (U, φ), on M, there is a bijection

$$\tau_U\colon \pi^{-1}(U) \to U \times \mathbb{R}^n,$$

given by

$$\tau_U(v) = (\pi(v), \theta^{-1}_{U,\varphi,\pi(v)}(v)).$$

Clearly, $pr_1 \circ \tau_U = \pi$, on $\pi^{-1}(U)$.

Thus, locally, that is, over U, the bundle $T(M)$ looks like the product $U \times \mathbb{R}^n$.

We say that $T(M)$ is *locally trivial* (over U) and we call τ_U a *trivializing map*.

For any $p \in M$, the vector space $\pi^{-1}(p) = T_p(M)$ is called the *fibre above p*.

Observe that the restriction of τ_U to $\pi^{-1}(p)$ is an isomorphism between $T_p(M)$ and $\{p\} \times \mathbb{R}^n \cong \mathbb{R}^n$, for any $p \in M$.

All these ingredients are part of being a *vector bundle* (but a little more is required of the maps τ_U). For more on bundles, see Lang [**?**], Gallot, Hulin and Lafontaine [**?**], Lafontaine [**?**] or Bott and Tu [**?**].

When $M = \mathbb{R}^n$, observe that
$T(M) = M \times \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, i.e., the bundle $T(M)$ is (globally) trivial.

Given a C^k-map, $h\colon M \to N$, between two C^k-manifolds, we can define the function, $dh\colon T(M) \to T(N)$, (also denoted Th, or h_*, or Dh) by setting

$$dh(u) = dh_p(u), \quad \text{iff} \quad u \in T_p(M).$$

We leave the next proposition as an exercise to the reader (A proof can be found in Berger and Gostiaux [**?**]).

Proposition 6.3.1 *Given a C^k-map, $h: M \to N$, between two C^k-manifolds M and N (with $k \geq 1$), the map $dh: T(M) \to T(N)$ is a C^{k-1}-map.*

We are now ready to define vector fields.

Definition 6.3.2 Let M be a C^{k+1} manifold, with $k \geq 1$. For any open subset, U of M, a *vector field on* U is any *section, ξ, of $T(M)$ over U*, i.e., any function, $\xi: U \to T(M)$, such that $\pi \circ \xi = \mathrm{id}_U$ (i.e., $\xi(p) \in T_p(M)$, for every $p \in U$). We also say that ξ is a *lifting of U into $T(M)$*.

We say that ξ is a *C^h-vector field on U* iff ξ is a section over U and a C^h-map, where $0 \leq h \leq k$.

The set of C^k-vector fields over U is denoted $\Gamma^{(k)}(U, T(M))$. Given a curve, $\gamma: [a, b] \to M$, a *vector field, ξ, along* γ is any section of $T(M)$ over γ, i.e., a C^k-function, $\xi: [a, b] \to T(M)$, such that $\pi \circ \xi = \gamma$. We also say that ξ *lifts γ into $T(M)$*.

The above definition gives a precise meaning to the idea that a C^k-vector field on M is an assignment, $p \mapsto \xi(p)$, of a tangent vector, $\xi(p) \in T_p(M)$, to a point, $p \in M$, so that $\xi(p)$ varies in a C^k-fashion in terms of p.

Clearly, $\Gamma^{(k)}(U, T(M))$ is a real vector space. For short, the space $\Gamma^{(k)}(M, T(M))$ is also denoted by $\Gamma^{(k)}(T(M))$ (or $\mathfrak{X}^{(k)}(M)$ or even $\Gamma(T(M))$ or $\mathfrak{X}(M)$).

If $M = \mathbb{R}^n$ and U is an open subset of M, then $T(M) = \mathbb{R}^n \times \mathbb{R}^n$ and a section of $T(M)$ over U is simply a function, ξ, such that

$$\xi(p) = (p, u), \quad \text{with} \quad u \in \mathbb{R}^n,$$

for all $p \in U$. In other words, ξ is defined by a function, $f: U \to \mathbb{R}^n$ (namely, $f(p) = u$).

This corresponds to the "old" definition of a vector field in the more basic case where the manifold, M, is just $\mathbb{R}^n$.

Given any C^k-function, $f \in \mathcal{C}^k(U)$, and a vector field, $\xi \in \Gamma^{(k)}(U, T(M))$, we define the vector field, $f\xi$, by

$$(f\xi)(p) = f(p)\xi(p), \quad p \in U.$$

Obviously, $f\xi \in \Gamma^{(k)}(U, T(M))$, which shows that $\Gamma^{(k)}(U, T(M))$ is also a $\mathcal{C}^k(U)$-module. We also denote $\xi(p)$ by ξ_p.

For any chart, (U, φ), on M it is easy to check that the map

$$p \mapsto \left(\frac{\partial}{\partial x_i}\right)_p, \quad p \in U,$$

is a C^k-vector field on U (with $1 \leq i \leq n$). This vector field is denoted $\left(\frac{\partial}{\partial x_i}\right)$ or $\frac{\partial}{\partial x_i}$.

If U is any open subset of M and f is any function in $\mathcal{C}^k(U)$, then $\xi(f)$ is the function on U given by

$$\xi(f)(p) = \xi_p(f) = \xi_p(\mathbf{f}).$$

As a special case, when (U, φ) is a chart on M, the vector field, $\frac{\partial}{\partial x_i}$, just defined above induces the function

$$p \mapsto \left(\frac{\partial}{\partial x_i}\right)_p f, \quad p \in U,$$

denoted $\frac{\partial}{\partial x_i}(f)$ or $\left(\frac{\partial}{\partial x_i}\right) f$.

It is easy to check that $\xi(f) \in \mathcal{C}^{k-1}(U)$. As a consequence, every vector field $\xi \in \Gamma^{(k)}(U, T(M))$ induces a linear map,

$$L_\xi \colon \mathcal{C}^k(U) \longrightarrow \mathcal{C}^{k-1}(U),$$

given by $f \mapsto \xi(f)$.

It is immediate to check that L_ξ has the Leibnitz property, i.e.,

$$L_\xi(fg) = L_\xi(f)g + fL_\xi(g).$$

Linear maps with this property are called *derivations*.

Thus, we see that every vector field induces some kind of differential operator, namely, a linear derivation.

Unfortunately, not every linear derivation of the above type arises from a vector field, although this turns out to be true in the smooth case i.e., when $k = \infty$ (for a proof, see Gallot, Hulin and Lafontaine [**?**] or Lafontaine [**?**]).

In the rest of this section, unless stated otherwise, we assume that $k \geq 1$. The following easy proposition holds (c.f. Warner [**?**]):

Proposition 6.3.3 *Let ξ be a vector field on the C^{k+1}-manifold, M, of dimension n. Then, the following are equivalent:*

(a) ξ is C^k.

(b) If (U, φ) is a chart on M and if $f_1, \ldots, f_n$ are the functions on U uniquely defined by

$$\xi \restriction U = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i},$$

then each f_i is a C^k-map.

(c) Whenever U is open in M and $f \in \mathcal{C}^k(U)$, then $\xi(f) \in \mathcal{C}^{k-1}(U)$.

Given any two C^k-vector field, ξ, η, on M, for any function, $f \in \mathcal{C}^k(M)$, we defined above the function $\xi(f)$ and $\eta(f)$.

Thus, we can form $\xi(\eta(f))$ (resp. $\eta(\xi(f))$), which are in $\mathcal{C}^{k-2}(M)$.

Unfortunately, even in the smooth case, there is generally no vector field, ζ, such that

$$\zeta(f) = \xi(\eta(f)), \quad \text{for all } f \in \mathcal{C}^k(M).$$

This is because $\xi(\eta(f))$ (and $\eta(\xi(f))$) involve second-order derivatives.

However, if we consider $\xi(\eta(f)) - \eta(\xi(f))$, then second-order derivatives cancel out and there is a unique vector field inducing the above differential operator.

Intuitively, $\xi\eta - \eta\xi$ measures the "failure of ξ and η to commute".

Proposition 6.3.4 *Given any C^{k+1}-manifold, M, of dimension n, for any two C^k-vector fields, ξ, η, on M, there is a unique C^{k-1}-vector field, $[\xi, \eta]$, such that*

$$[\xi, \eta](f) = \xi(\eta(f)) - \eta(\xi(f)), \quad \textit{for all} \quad f \in \mathcal{C}^{k-1}(M).$$

Definition 6.3.5 Given any C^{k+1}-manifold, M, of dimension n, for any two C^k-vector fields, ξ, η, on M, the *Lie bracket*, $[\xi, \eta]$, of ξ and η, is the C^{k-1} vector field defined so that

$$[\xi, \eta](f) = \xi(\eta(f)) - \eta(\xi(f)), \quad \text{for all} \quad f \in \mathcal{C}^{k-1}(M).$$

We also have the following simple proposition whose proof is left as an exercise (or, see Do Carmo [**?**]):

Proposition 6.3.6 *Given any C^{k+1}-manifold, M, of dimension n, for any C^k-vector fields, ξ, η, ζ, on M, for all $f, g \in \mathcal{C}^k(M)$, we have:*

(a) $[[\xi, \eta], \zeta] + [[\eta, \zeta], \xi] + [[\zeta, \xi], \eta] = 0$ *(Jacobi identity).*

(b) $[\xi, \xi] = 0$.

(c) $[f\xi, g\eta] = fg[\xi, \eta] + f\xi(g)\eta - g\eta(f)\xi$.

(d) $[-, -]$ *is bilinear.*

Consequently, for smooth manifolds ($k = \infty$), the space of vector fields, $\Gamma^{(\infty)}(T(M))$, is a vector space equipped with a bilinear operation, $[-, -]$, that satisfies the Jacobi identity.

This makes $\Gamma^{(\infty)}(T(M))$ a *Lie algebra*.

One more notion will be needed when we deal with Lie algebras.

Definition 6.3.7 Let $\varphi\colon M \to N$ be a C^{k+1}-map of manifolds. If ξ is a C^k vector field on M and η is a C^k vector field on N, we say that *ξ and η are φ-related* iff

$$d\varphi \circ \xi = \eta \circ \varphi.$$

Proposition 6.3.8 *Let $\varphi\colon M \to N$ be a C^{k+1}-map of manifolds, let ξ and ξ_1 be C^k vector fields on M and let η, η_1 be C^k vector fields on N. If ξ is φ-related to ξ_1 and η is φ-related to η_1, then $[\xi, \eta]$ is φ-related to $[\xi_1, \eta_1]$.*

6.4 Submanifolds, Immersions, Embeddings

Although the notion of submanifold is intuitively rather clear, technically, it is a bit tricky.

In fact, the reader may have noticed that many different definitions appear in books and that it is not obvious at first glance that these definitions are equivalent.

What is important is that a submanifold, N of a given manifold, M, not only have the topology induced M but also that the charts of N be somewhow induced by those of M.

(Recall that if X is a topological space and Y is a subset of X, then the *subspace topology on* Y or *topology induced by* X *on* Y has for its open sets all subsets of the form $Y \cap U$, where U is an arbitary subset of X.).

Given m, n, with $0 \leq m \leq n$, we can view $\mathbb{R}^m$ as a subspace of $\mathbb{R}^n$ using the inclusion

$$\mathbb{R}^m \cong \mathbb{R}^m \times \{\underbrace{(0, \ldots, 0)}_{n-m}\} \hookrightarrow \mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n,$$

given by

$$(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, \underbrace{0, \ldots, 0}_{n-m}).$$

Definition 6.4.1 Given a C^k-manifold, M, of dimension n, a subset, N, of M is an *m-dimensional submanifold of M* (where $0 \leq m \leq n$) iff for every point, $p \in N$, there is a chart, (U, φ), of M, with $p \in U$, so that

$$\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^m \times \{0_{n-m}\}).$$

(We write $0_{n-m} = \underbrace{(0, \ldots, 0)}_{n-m}$.)

The subset, $U \cap N$, of Definition 6.4.1 is sometimes called a *slice* of (U, φ) and we say that (U, φ) is *adapted to N* (See O'Neill [**?**] or Warner [**?**]).

Other authors, including Warner [**?**], use the term submanifold in a broader sense than us and they use the word *embedded submanifold* for what is defined in Definition 6.4.1.

The following proposition has an almost trivial proof but it justifies the use of the word submanifold:

Proposition 6.4.2 *Given a C^k-manifold, M, of dimension n, for any submanifold, N, of M of dimension $m \leq n$, the family of pairs $(U \cap N, \varphi \restriction U \cap N)$, where (U, φ) ranges over the charts over any atlas for M, is an atlas for N, where N is given the subspace topology. Therefore, N inherits the structure of a C^k-manifold.*

In fact, every chart on N arises from a chart on M in the following precise sense:

Proposition 6.4.3 *Given a C^k-manifold, M, of dimension n and a submanifold, N, of M of dimension $m \leq n$, for any $p \in N$ and any chart, (W, η), of N at p, there is some chart, (U, φ), of M at p so that*

$$\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^m \times \{0_{n-m}\})$$

and

$$\varphi \restriction U \cap N = \eta \restriction U \cap N,$$

where $p \in U \cap N \subseteq W$.

It is also useful to define more general kinds of "submanifolds".

Definition 6.4.4 Let $\varphi\colon N \to M$ be a C^k-map of manifolds.

(a) The map φ is an *immersion* of N into M iff $d\varphi_p$ is injective for all $p \in N$.

(b) The set $\varphi(N)$ is an *immersed submanifold* of M iff φ is an injective immersion.

(c) The map φ is an *embedding* of N into M iff φ is an injective immersion such that the induced map, $N \longrightarrow \varphi(N)$, is a homeomorphism, where $\varphi(N)$ is given the subspace topology (equivalently, φ is an open map from N into $\varphi(N)$ with the subspace topology). We say that $\varphi(N)$ (with the subspace topology) is an *embedded submanifold* of M.

(d) The map φ is a *submersion* of N into M iff $d\varphi_p$ is surjective for all $p \in N$.

Again, we warn our readers that certain authors (such as Warner [**?**]) call $\varphi(N)$, in (b), a submanifold of M! We prefer the terminology *immersed submanifold*.

The notion of immersed submanifold arises naturally in the framewok of Lie groups.

Indeed, the fundamental correspondence between Lie groups and Lie algebras involves Lie subgroups that are not necessarily closed.

But, as we will see later, subgroups of Lie groups that are also submanifolds are always closed.

It is thus necessary to have a more inclusive notion of submanifold for Lie groups and the concept of immersed submanifold is just what's needed.

Immersions of $\mathbb{R}$ into $\mathbb{R}^3$ are parametric curves and immersions of $\mathbb{R}^2$ into $\mathbb{R}^3$ are parametric surfaces. These have been extensively studied, for example, see DoCarmo [**?**], Berger and Gostiaux [**?**] or Gallier [**?**].

Immersions (i.e., subsets of the form $\varphi(N)$, where N is an immersion) are generally neither injective immersions (i.e., subsets of the form $\varphi(N)$, where N is an injective immersion) nor embeddings (or submanifolds).

For example, immersions can have self-intersections, as the plane curve (nodal cubic): $x = t^2 - 1; y = t(t^2 - 1)$.

Injective immersions are generally not embeddings (or submanifolds) because $\varphi(N)$ may not be homeomorphic to N.

An example is given by the Lemniscate of Bernoulli, an injective immersion of $\mathbb{R}$ into $\mathbb{R}^2$:

$$\begin{aligned} x &= \frac{t(1+t^2)}{1+t^4}, \\ y &= \frac{t(1-t^2)}{1+t^4}. \end{aligned}$$

There is, however, a close relationship between submanifolds and embeddings.

Proposition 6.4.5 *If N is a submanifold of M, then the inclusion map, $j\colon N \to M$, is an embedding. Conversely, if $\varphi\colon N \to M$ is an embedding, then $\varphi(N)$ with the subspace topology is a submanifold of M and φ is a diffeomorphism between N and $\varphi(N)$.*

In summary, embedded submanifolds and (our) submanifolds coincide.

Some authors refer to spaces of the form $\varphi(N)$, where φ is an injective immersion, as *immersed submanifolds*.

However, in general, an immersed submanifold is *not* a submanifold.

One case where this holds is when N is compact, since then, a bijective continuous map is a homeomorphism.

Our next goal is to review and promote to manifolds some standard results about ordinary differential equations.

6.5 Integral Curves, Flow of a Vector Field, One-Parameter Groups of Diffeomorphisms

We begin with integral curves and (local) flows of vector fields on a manifold.

Definition 6.5.1 Let ξ be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$ and let p_0 be a point on M. An *integral curve (or trajectory) for* ξ *with initial condition* p_0 is a C^{p-1}-curve, $\gamma\colon I \to M$, so that

$$\dot{\gamma}(t) = \xi(\gamma(t)), \quad \text{for all } t \in I \quad \text{and} \quad \gamma(0) = p_0,$$

where $I = \,]a, b[\, \subseteq \mathbb{R}$ is an open interval containing 0.

What definition 6.5.1 says is that an integral curve, γ, with initial condition p_0 is a curve on the manifold M passing through p_0 and such that, for every point $p = \gamma(t)$ on this curve, the tangent vector to this curve at p, i.e., $\dot{\gamma}(t)$, coincides with the value, $\xi(p)$, of the vector field ξ at p.

Given a vector field, ξ, as above, and a point $p_0 \in M$, is there an integral curve through p_0? Is such a curve unique? If so, how large is the open interval I?

We provide some answers to the above questions below.

Definition 6.5.2 Let ξ be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$ and let p_0 be a point on M. A *local flow for ξ at p_0* is a map,

$$\varphi\colon J \times U \to M,$$

where $J \subseteq \mathbb{R}$ is an open interval containing 0 and U is an open subset of M containing p_0, so that for every $p \in U$, the curve $t \mapsto \varphi(t, p)$ is an integral curve of ξ with initial condition p.

Thus, a local flow for ξ is a family of integral curves for all points in some small open set around p_0 such that these curves all have the same domain, J, independently of the initial condition, $p \in U$.

The following theorem is the main existence theorem of local flows.

This is a promoted version of a similar theorem in the classical theory of ODE's in the case where M is an open subset of $\mathbb{R}^n$.

Theorem 6.5.3 *(Existence of a local flow) Let ξ be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$ and let p_0 be a point on M. There is an open interval, $J \subseteq \mathbb{R}$, containing 0 and an open subset, $U \subseteq M$, containing p_0, so that there is a* **unique** *local flow, $\varphi\colon J \times U \to M$, for ξ at p_0. Furthermore, φ is C^{k-1}.*

Theorem 6.5.3 holds under more general hypotheses, namely, when the vector field satisfies some *Lipschitz* condition, see Lang [**?**] or Berger and Gostiaux [**?**].

Now, we know that for any initial condition, p_0, there is some integral curve through p_0.

However, there could be two (or more) integral curves $\gamma_1\colon I_1 \to M$ and $\gamma_2\colon I_2 \to M$ with initial condition p_0.

This leads to the natural question: How do γ_1 and γ_2 differ on $I_1 \cap I_2$? The next proposition shows they don't!

Proposition 6.5.4 *Let ξ be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$ and let p_0 be a point on M. If $\gamma_1\colon I_1 \to M$ and $\gamma_2\colon I_2 \to M$ are any two integral curves both with initial condition p_0, then $\gamma_1 = \gamma_2$ on $I_1 \cap I_2$.*

Proposition 6.5.4 implies the important fact that there is a *unique maximal* integral curve with initial condition p.

Indeed, if $\{\gamma_k\colon I_k \to M\}_{k\in K}$ is the family of all integral curves with initial condition p (for some big index set, K), if we let $I(p) = \bigcup_{k\in K} I_k$, we can define a curve, $\gamma_p\colon I(p) \to M$, so that

$$\gamma_p(t) = \gamma_k(t), \quad \text{if} \quad t \in I_k.$$

Since γ_k and γ_l agree on $I_k \cap I_l$ for all $k, l \in K$, the curve γ_p is indeed well defined and it is clearly an integral curve with initial condition p with the largest possible domain (the open interval, $I(p)$).

The curve γ_p is called the *maximal integral curve with initial condition* p and it is also denoted $\gamma(t, p)$.

Note that Proposition 6.5.4 implies that any two distinct integral curves are disjoint, i.e., do not intersect each other.

The following interesting question now arises: Given any $p_0 \in M$, if $\gamma_{p_0} \colon I(p_0) \to M$ is the maximal integral curve with initial condition p_0, for any $t_1 \in I(p_0)$, and if $p_1 = \gamma_{p_0}(t_1) \in M$, then there is a maximal integral curve, $\gamma_{p_1} \colon I(p_1) \to M$, with initial condition p_1.

What is the relationship between γ_{p_0} and γ_{p_1}, if any? The answer is given by

Proposition 6.5.5 *Let ξ be a C^{k-1} vector field on a C^k-manifold, M, ($k \geq 2$) and let p_0 be a point on M. If $\gamma_{p_0}\colon I(p_0) \to M$ is the maximal integral curve with initial condition p_0, for any $t_1 \in I(p_0)$, if $p_1 = \gamma_{p_0}(t_1) \in M$ and $\gamma_{p_1}\colon I(p_1) \to M$ is the maximal integral curve with initial condition p_1, then*

$$I(p_1) = I(p_0) - t_1 \quad \textit{and} \quad \gamma_{p_1}(t) = \gamma_{\gamma_{p_0}(t_1)}(t) = \gamma_{p_0}(t+t_1),$$

for all $t \in I(p_0) - t_1$

It is useful to restate Proposition 6.5.5 by changing point of view.

So far, we have been focusing on integral curves, i.e., given any $p_0 \in M$, we let t vary in $I(p_0)$ and get an integral curve, γ_{p_0}, with domain $I(p_0)$.

Instead of holding $p_0 \in M$ fixed, we can hold $t \in \mathbb{R}$ fixed and consider the set

$$\mathcal{D}_t(\xi) = \{p \in M \mid t \in I(p)\},$$

i.e., the set of points such that it is possible to "travel for t units of time from p" along the maximal integral curve, γ_p, with initial condition p (It is possible that $\mathcal{D}_t(\xi) = \emptyset$).

By definition, if $\mathcal{D}_t(\xi) \neq \emptyset$, the point $\gamma_p(t)$ is well defined, and so, we obtain a map,
$\Phi_t^\xi \colon \mathcal{D}_t(\xi) \to M$, with domain $\mathcal{D}_t(\xi)$, given by

$$\Phi_t^\xi(p) = \gamma_p(t).$$

The above suggests the following definition:

Definition 6.5.6 Let ξ be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$. For any $t \in \mathbb{R}$, let

$$\mathcal{D}_t(\xi) = \{p \in M \mid t \in I(p)\}$$

and

$$\mathcal{D}(\xi) = \{(t, p) \in \mathbb{R} \times M \mid t \in I(p)\}$$

and let $\Phi^\xi\colon \mathcal{D}(\xi) \to M$ be the map given by

$$\Phi^\xi(t, p) = \gamma_p(t).$$

The map Φ^ξ is called the *(global) flow of* ξ and $\mathcal{D}(\xi)$ is called its *domain of definition*. For any $t \in \mathbb{R}$ such that $\mathcal{D}_t(\xi) \neq \emptyset$, the map, $p \in \mathcal{D}_t(\xi) \mapsto \Phi^\xi(t, p) = \gamma_p(t)$, is denoted by Φ_t^ξ (i.e., $\Phi_t^\xi(p) = \Phi^\xi(t, p) = \gamma_p(t)$).

Observe that

$$\mathcal{D}(\xi) = \bigcup_{p \in M} (I(p) \times \{p\}).$$

Also, using the Φ_t^ξ notation, the property of Proposition 6.5.5 reads

$$\Phi_s^\xi \circ \Phi_t^\xi = \Phi_{s+t}^\xi, \tag{$*$}$$

whenever both sides of the equation make sense.

Indeed, the above says

$$\Phi_s^\xi(\Phi_t^\xi(p)) = \Phi_s^\xi(\gamma_p(t)) = \gamma_{\gamma_p(t)}(s) = \gamma_p(s+t) = \Phi_{s+t}^\xi(p).$$

Using the above property, we can easily show that the Φ_t^ξ are invertible. In fact, the inverse of Φ_t^ξ is Φ_{-t}^ξ.

We summarize in the following proposition some additional properties of the domains $\mathcal{D}(\xi)$, $\mathcal{D}_t(\xi)$ and the maps Φ_t^ξ:

Theorem 6.5.7 *Let ξ be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$. The following properties hold:*

(a) For every $t \in \mathbb{R}$, if $\mathcal{D}_t(\xi) \neq \emptyset$, then $\mathcal{D}_t(\xi)$ is open (this is trivially true if $\mathcal{D}_t(\xi) = \emptyset$).

(b) The domain, $\mathcal{D}(\xi)$, of the flow, Φ^ξ, is open and the flow is a C^{k-1} map, $\Phi^\xi \colon \mathcal{D}(\xi) \to M$.

(c) Each $\Phi^\xi_t \colon \mathcal{D}_t(\xi) \to \mathcal{D}_{-t}(\xi)$ is a C^{k-1}-diffeomorphism with inverse Φ^ξ_{-t}.

(d) For all $s, t \in \mathbb{R}$, the domain of definition of $\Phi^\xi_s \circ \Phi^\xi_t$ is contained but generally not equal to $\mathcal{D}_{s+t}(\xi)$. However, $\mathrm{dom}(\Phi^\xi_s \circ \Phi^\xi_t) = \mathcal{D}_{s+t}(\xi)$ if s and t have the same sign. Moreover, on $\mathrm{dom}(\Phi^\xi_s \circ \Phi^\xi_t)$, we have

$$\Phi^\xi_s \circ \Phi^\xi_t = \Phi^\xi_{s+t}.$$

The reason for using the terminology flow in referring to the map Φ^ξ can be clarified as follows:

For any t such that $\mathcal{D}_t(\xi) \neq \emptyset$, every integral curve, γ_p, with initial condition $p \in \mathcal{D}_t(\xi)$, is defined on some open interval containing $[0, t]$, and we can picture these curves as "flow lines" along which the points p flow (travel) for a time interval t.

Then, $\Phi^{\xi}(t, p)$ is the point reached by "flowing" for the amount of time t on the integral curve γ_p (through p) starting from p.

Intuitively, we can imagine the flow of a fluid through M, and the vector field ξ is the field of velocities of the flowing particles.

Given a vector field, ξ, as above, it may happen that $\mathcal{D}_t(\xi) = M$, for all $t \in \mathbb{R}$.

In this case, namely, when $\mathcal{D}(\xi) = \mathbb{R} \times M$, we say that the vector field ξ is *complete*.

Then, the Φ_t^ξ are diffeomorphisms of M and they form a group.

The family $\{\Phi_t^\xi\}_{t\in\mathbb{R}}$ a called a 1-*parameter group of* ξ.

In this case, Φ^ξ induces a group homomorphism, $(\mathbb{R}, +) \longrightarrow \mathrm{Diff}(M)$, from the additive group $\mathbb{R}$ to the group of C^{k-1}-diffeomorphisms of M.

By abuse of language, even when it is **not** the case that $\mathcal{D}_t(\xi) = M$ for all t, the family $\{\Phi_t^\xi\}_{t\in\mathbb{R}}$ is called a *local* 1-*parameter group of* ξ, even though it is **not** a group, because the composition $\Phi_s^\xi \circ \Phi_t^\xi$ may not be defined.

When M is compact, it turns out that every vector field is complete, a nice and useful fact.

Proposition 6.5.8 *Let ξ be a C^{k-1} vector field on a C^k-manifold, M, ($k \geq 2$). If M is compact, then ξ is complete, i.e., $\mathcal{D}(\xi) = \mathbb{R} \times M$. Moreover, the map $t \mapsto \Phi_t^\xi$ is a homomorphism from the additive group $\mathbb{R}$ to the group, $\mathrm{Diff}(M)$, of (C^{k-1}) diffeomorphisms of M.*

Remark: The proof of Proposition 6.5.8 also applies when ξ is a vector field with compact support (this means that the closure of the set $\{p \in M \mid \xi(p) \neq 0\}$ is compact).

A point $p \in M$ where a vector field vanishes, i.e., $\xi(p) = 0$, is called a *critical point of* ξ.

Critical points play a major role in the study of vector fields, in differential topology (e.g., the celebrated Poincaré–Hopf index theorem) and especially in Morse theory, but we won't go into this here (curious readers should consult Milnor [**?**], Guillemin and Pollack [**?**] or DoCarmo [**?**], which contains an informal but very clear presentation of the Poincaré–Hopf index theorem).

Another famous theorem about vector fields says that every smooth vector field on a sphere of even dimension (S^{2n}) must vanish in at least one point (the so-called "hairy-ball theorem".

On S^2, it says that you can't comb your hair without having a singularity somewhere. Try it, it's true!).

Let us just observe that if an integral curve, γ, passes through a critical point, p, then γ is reduced to the point p, i.e., $\gamma(t) = p$, for all t.

Then, we see that if a maximal integral curve is defined on the whole of $\mathbb{R}$, either it is injective (it has no self-intersection), or it is simply periodic (i.e., there is some $T > 0$ so that $\gamma(t + T) = \gamma(t)$, for all $t \in \mathbb{R}$ and γ is injective on $[0, T[$), or it is reduced to a single point.

We conclude this section with the definition of the Lie derivative of a vector field with respect to another vector field.

Say we have two vector fields ξ and η on M. For any $p \in M$, we can flow along the integral curve of ξ with initial condition p to $\Phi_t^{\xi}(p)$ (for t small enough) and then evaluate η there, getting $\eta(\Phi_t^{\xi}(p))$.

Now, this vector belongs to the tangent space $T_{\Phi_t^{\xi}(p)}(M)$, but $\eta(p) \in T_p(M)$.

So to "compare" $\eta(\Phi_t^{\xi}(p))$ and $\eta(p)$, we bring back $\eta(\Phi_t^{\xi}(p))$ to $T_p(M)$ by applying the tangent map, $d\Phi_{-t}^{\xi}$, at $\Phi_t^{\xi}(p)$, to $\eta(\Phi_t^{\xi}(p))$ (Note that to alleviate the notation, we use the slight abuse of notation $d\Phi_{-t}^{\xi}$ instead of $d(\Phi_{-t}^{\xi})_{\Phi_t^{\xi}(p)}$.)

Then, we can form the difference $d\Phi^{\xi}_{-t}(\eta(\Phi^{\xi}_{t}(p))) - \eta(p)$, divide by t and consider the limit as t goes to 0.

This is the *Lie derivative of* η *with respect to* ξ *at* p, denoted $(L_{\xi}\,\eta)_p$, and given by

$$(L_{\xi}\,\eta)_p = \lim_{t\longrightarrow 0} \frac{d\Phi^{\xi}_{-t}(\eta(\Phi^{\xi}_{t}(p))) - \eta(p)}{t} = \frac{d}{dt}\left(d\Phi^{\xi}_{-t}(\eta(\Phi^{\xi}_{t}(p)))\right)\bigg|_{t=0}.$$

It can be shown that $(L_{\xi}\,\eta)_p$ is our old friend, the Lie bracket, i.e.,

$$(L_{\xi}\,\eta)_p = [\xi, \eta]_p.$$