A SCAFFOLDED APPROACH TO INTRODUCING THE DEFINITION OF CONVERGENCE OF A SEQUENCE: FROM CONCEPT IMAGE TO CONCEPT DEFINITION

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Abstract. We describe a method for introducing the definition of convergence for a sequence, usually encountered in a first undergraduate course on analysis. This approach allows the students to understand the purpose of the various quantifiers and their order in the definition.

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1. Introduction

Few would argue that proofs are one of the most important aspects of modern mathematics and consequently in undergraduate mathematics much of students' time is spent on practicing reading, writing and understanding these. This however, is also one of the most challenging aspects of learning mathematics, as much of our shared experience teaching undergraduate courses tells us!

One of the difficulties in learning to produce formal proofs is the fact that *definitions* must be used in a very precise way and so there has been a corresponding focus on how students understand, interpret and learn to use precise formal definitions [1].

Arguably the first time students will see a definition that is sufficiently complex that it cannot be understood without either experience or concerted effort is in an analysis course that deals with convergence and limits. Typically the definition of convergence for a sequence of real numbers takes this role.

As Davis and Vinner pointed out [2] (see also [9]), part of the problem that students have reasoning with the definition is that they have conflicting ideas coming from prior experience: even though they may not have encountered the concept explicitly before they hold preconceived concept images that are incompatible with the concept definition and they may use these (or a mixture of the two, once they have learned the formal definition) to reason in any given instance.

The reason that the definition of this particular concept is so much more difficult lies in the quantifiers needed to express it formally. The struggle that students typically face when dealing with these was illustrated by Dubinsky, Elterman and Gong [4], who analysed the processes students used when attempting to understand statements with several quantifiers.

On the other hand, there has also been research into the actual *act* of defining mathematical objects [10], i.e. using the act of defining concepts or objects as an actual activity. For example, Edwards and Ward [5] state that: "Definition activities in mathematics courses can have several pedagogical objectives, some of which could be

- promoting deeper conceptual understanding of the mathematics involved,
- promoting an understanding of the nature or the characteristics of mathematical definitions, and/or
- promoting an understanding of the role of definitions in mathematics."

All of these are of course worthwhile learning outcomes for students in an undergraduate course.

Studies that investigated the outcomes of having students attempt to recreate definitions in analysis include [3, 8]. Kyong also describes classroom activities that guide students through the defining process [7].

In this paper we describe a new series of activities that lead students to construct the formal definition of convergence of a sequence, beginning with only their concept image, formed from previous experience and education.

2. Constructing the definition of convergence

The aim of the activities described in this paper is to allow students with no prior exposure to the formal definition of convergence of a sequence of real numbers to understand the definition by progressively constructing the various components with quantifiers attached to them.

For this purpose, we use the following definition of convergence of a sequence: A sequence (a_n) converges to L if

$$\forall\,\epsilon > 0\ \exists\,N \in \mathbb{N} \text{ such that } \forall\,n \geq N,\ |a_n - L| < \epsilon.$$

One remark worth making is that it is common to rewrite the last part as an implication: $n \geq N \implies |a_n - L| < \epsilon$. These are of course logically equivalent but for our purposes explicitly displaying all of the quantifiers is helpful because this is how we will build the definition in the sequel.

One of the crucial aspects of learning the definition of convergence—and in constructing rigorous proofs—is understanding the role of each of the quantifiers involved and so at each stage we ask the students to write down formally what they have described. In order to smooth this process we aim to make the descriptions as precise as possible before attempting to write down the accompanying formal mathematical statements.

We will essentially be building the definition by successive approximation: constructing "working definitions" that acceptably describe various examples and then adding in one quantifier at a time as we identify deficiencies until our working definition matches the actual definition.

2.1. **Preliminaries: agreeing on a concept image.** The first stage in the creation of the formal definition of convergence is for the students to have as closely as possible matching intuition or concept images for convergence. Therefore, we begin by asking students to identify sequences that converge and sequences that do not (without knowing precisely what it means to converge). This allows us to (1) ensure that everyone is in (rough) agreement to begin with and (2) identify any misconceptions that students may have about the concept before attempting a rigorous definition.

To accomplish this, we give some graphical examples of sequences (this way the students are not tempted to focus on calculational or procedural methods and are essentially forced to engage on a conceptual level). The students are asked to decide whether each example "converges", for whatever notion of convergence they have in their mind.

Figures, 1 and 2 show sequences that students more-or-less unanimously agree converge. The second is slightly trickier since it is not monotonic, which students sometimes see as a necessary condition for convergence (see [2]). Figures 3 and 4 show sequences that do not converge. Most students agree that the sequence shown in Figure 3 doesn't converge, although sometimes there is debate about whether or not it is possible (i.e. whether or not we should allow the possibility in our definition) for a sequence to converge to more than one number. Figure 4 is the most problematic since it appears to be "converging" to the dashed line shown. Experience shows however, that students who are not trapped by this example will be able to easily point out to their peers that a sequence should only be allowed to converge to a single number, rather than a function.

2.2. **The first quantifier.** Once the students have agreed on a common concept image, they can begin to distill the crucial parts of this into a formal definition. This almost certainly requires some intervention and guidance (depending on how long you might have to discuss the topic!).

To begin with, the students are asked to identify the characteristics that the first two examples from Figures 1 and 2 have in common. Distilling a common theme is possibly the most difficult part for students and since we are trying to build the definition up in pieces it is useful to provide more guidance here than elsewhere. Therefore, it is helpful

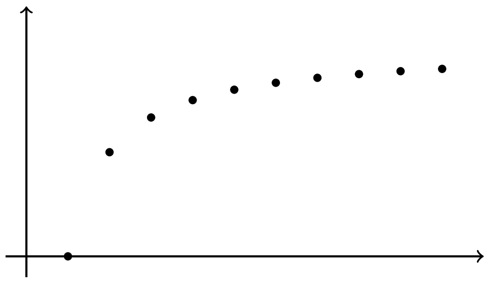

FIGURE 1. A clearly convergent sequence

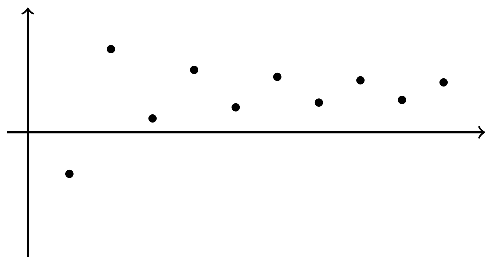

FIGURE 2. A clearly convergent sequence

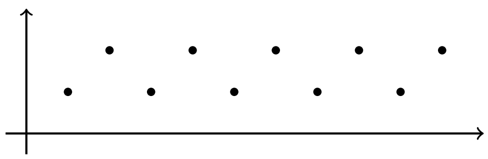

FIGURE 3. A "clearly" not convergent sequence

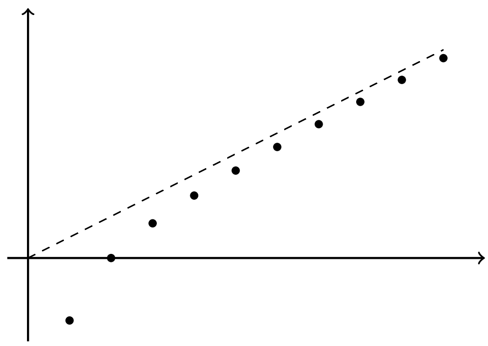

FIGURE 4. A not-so-clear example

to give them the following initial ideas: To decide if a sequence converges we should be following these steps:

(1) choose a (small) distance ϵ;

(2) ask: does the sequence get to within ϵ of some (particular, fixed) number L?

Now the students should attempt to write down a formal logical statement that is equivalent to the steps above. For now it is useful to fix ϵ and ask them to write down what the second point means.

The aim is for them to arrive at a statement as close as possible to

$$\exists\, N \in \mathbb{N} \text{ such that } |a_N - L| < \epsilon$$

which we take as our starting point for the next part of the exercise. Note that this starting point is in agreement with [6], who points out that the inequality $|a_n - L| < \epsilon$ is really at the heart of the definition.

2.3. **The second quantifier.** The description above is essentially deficient in two ways: firstly, the fact that ϵ must be arbitrary, and secondly that there is no requirement on the large n behaviour of the sequence.

We tackle the second of these first, since it is much easier to both observe and to fix (understanding the role that ϵ plays is perhaps the most difficult part of the process).

The students are asked about a sequence that satisfies the working definition from section 2.2 but does not converge. Figure 5 gives such an example and the students should (given their agreement about the sequence from Figure 3) decide that it does not converge. The crucial part is explaining *why* it does not.

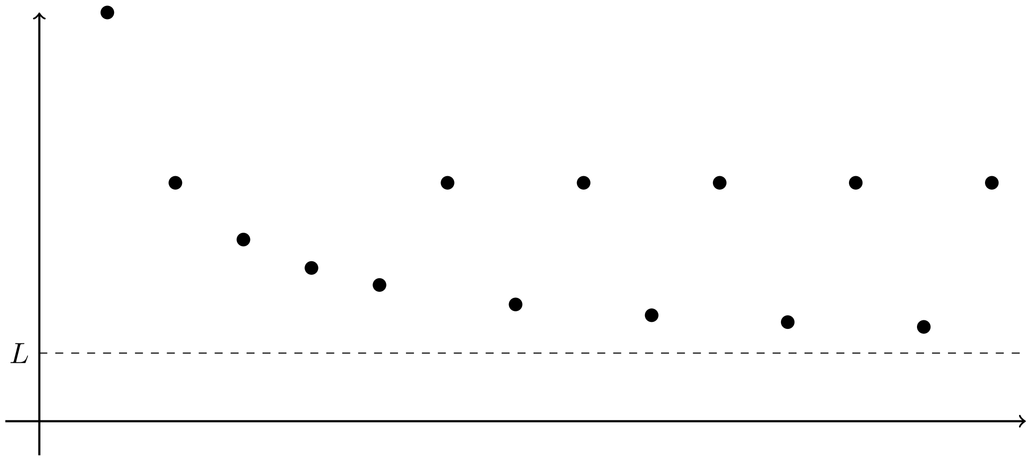

FIGURE 5. A sequence such that $\exists\, N$ s.t. $|a_N - L| < \epsilon$ but not for every $n \geq N$

Depending on how quickly the students arrive at a correctly (and precisely described) answer, this can be expedited by considering sequences as depicted in Figures 6 and 7.

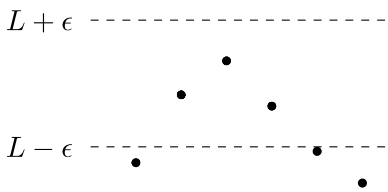

FIGURE 6. A sequence that enters and leaves the ϵ-strip

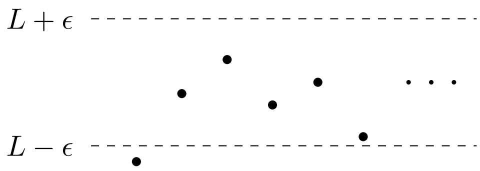

FIGURE 7. A sequence that enters and does not leave the ϵ-strip, where the $\cdots$ indicates that the sequence continues inside the strip shown

Once the students have identified the difference between the sequences depicted they should attempt to write down formally what this means.

The aim is for them to arrive at the statement with the additional quantifier:

$$\exists\, N \in \mathbb{N} \text{ such that } \forall\, n \geq N,\ |a_n - L| < \epsilon$$

which we again feed into the final part of the exercise.

2.4. **The final quantifier.** The only remaining adjustment that needs to be made in order to produce the correct definition is the universal quantifier at the very beginning. In order for students to understand the necessity of this we simply need ask the question: what happens if someone has chosen a smaller distance ϵ?

Showing the students a picture similar to Figure 8 will give them the starting point for a discussion on this, with the aim being to identify that what they have constructed so far needs to be true for *every* (positive) ϵ (the distinction between $\forall\, \epsilon$, $\forall\, \epsilon > 0$ and $\forall\, \epsilon \geq 0$ is also an interesting discussion that may well come up here).

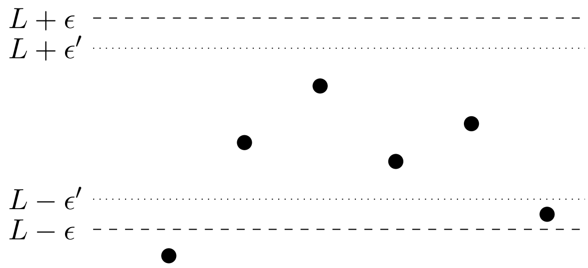

FIGURE 8. A sequence that demonstrates the necessity of allowing ϵ to be arbitrary

Finally, as before, once the students have agreed on the resolution to the problem (informally) they should be given the opportunity to write down precisely what they have decided the definition of convergence for a sequence should be.

3. Conclusion

We have outlined here a process that begins with students having no experience at all of the formal definition of convergence—and relying only on their intuition about what it should mean for a sequence to converge—and leads them through the process of (1) identifying precisely the important aspects of their intuition; and (2) turning this into formal mathematics.

We further conjecture that this process will also be useful later when students turn to proofs concerning limits and convergence since part of the process of writing a proof involves working out how to formalise the intuition about why the statement in question might be true.

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