

AN INTERACTIVE TOOL FOR VISUALISING EIGENVECTORS AND EIGENVALUES

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ABSTRACT. We present a novel, interactive app built using geogebra that allows students in undergraduate linear algebra courses to build a geometric understanding of eigenvectors and eigenvalues. The tool can be found at: <https://www.geogebra.org/m/rxuxegqp>

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1. INTRODUCTION

Many students studying linear algebra for the first time find it to be quite an abstract topic. For example, Harel [2] describes “concreteness” as one of three principles in the learning and teaching of linear algebra, which he contends is missing in many treatments. This point has also been noted for example in [3, 4], where an advocacy has been made for using techniques of visualisation in the teaching and learning of linear algebra. By the same notion, as pointed out in [1], traditional teaching approaches—i.e. transmitting information to students in a lecture format—are less effective in the teaching of linear algebra.

Therefore, it seems obvious that, in addition to active learning techniques (for example of the type described in [7] and pioneered by Mazur [5] we should provide students with

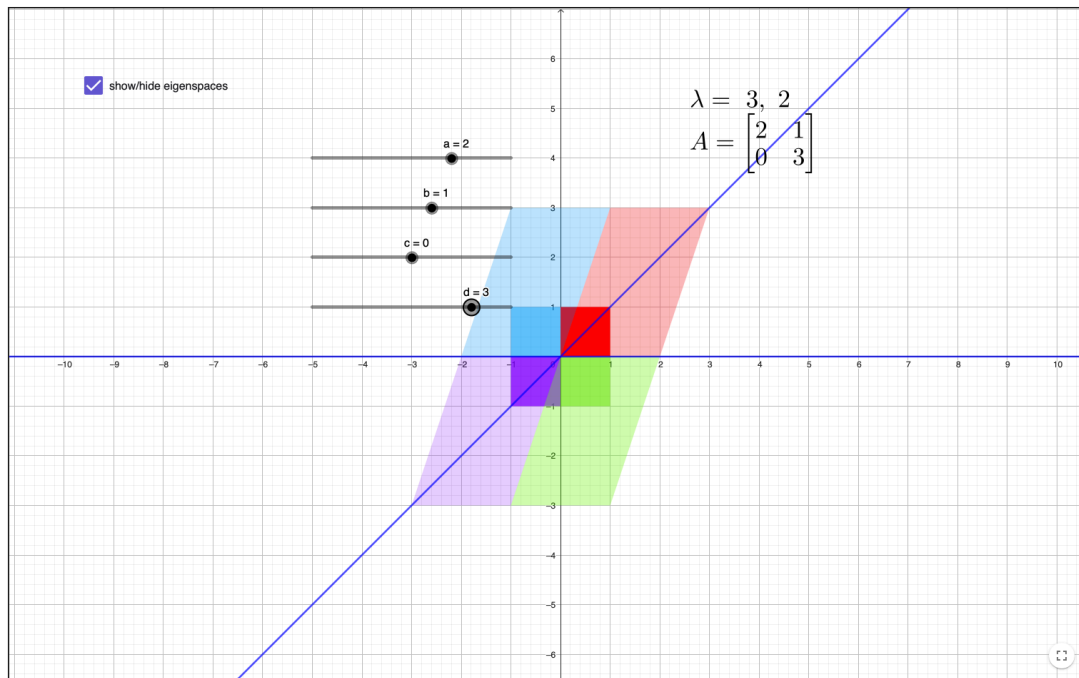


FIGURE 1. A geogebra app for visualising eigenvectors

alternative tools for visualising and understanding the abstract concepts encountered in linear algebra.

One of the most abstract—and important—of these concepts is that of eigenvalues and eigenvectors. The applications of these permeate almost every area of engineering, physics and mathematics, so it is clear that we would like students to form the best understanding of this topic that they can.

What we propose here is the use of a geometric tool, built using geogebra, that allows students to investigate the properties of eigenvectors and eigenvalues of 2×2 matrices.

Geogebra has been used as an effective learning tool in school level mathematics [6] and part of the purpose of this paper is to present an example of how it can be used effectively in tertiary level mathematics as well.

2. A GEOGEBRA APP FOR VISUALISING EIGENVECTORS AND EIGENVALUES

The purpose of the present paper is to introduce and demonstrate how the interactive web app [8] can be used to aid students in their understanding of eigenvectors and eigenvalues.

The app can be found here:

<https://www.geogebra.org/m/rxuxegqp>

Figure 1 shows the basic functionality of the app. The sliders on the left hand side vary the entries in the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The dark coloured square in the centre has corners $(\pm 1, \pm 1)$ and the lighter parallelogram shows the image of this square under the linear transformation whose standard matrix is A . The blue lines are the eigenspaces of A and the matrix and its eigenvalues are displayed in the top right. The eigenspaces can be toggled on and off in order to allow the user to either determine or to be shown them for various exercises or activities.

By varying the entries in the matrix A , we can see the effect on the eigenvalues and eigenspaces. The eigenspaces can also be identified by finding invariant lines in the image. In section 3 we will describe various results on eigenvectors and eigenvalues and give examples of how these can be investigated or understood using the app.

3. EXAMPLES OF INTERACTIVITY IN AN UNDERGRADUATE LINEAR ALGEBRA COURSE

In a typical course where students first encounter eigenvectors and eigenvalues they might learn about any of the following things:

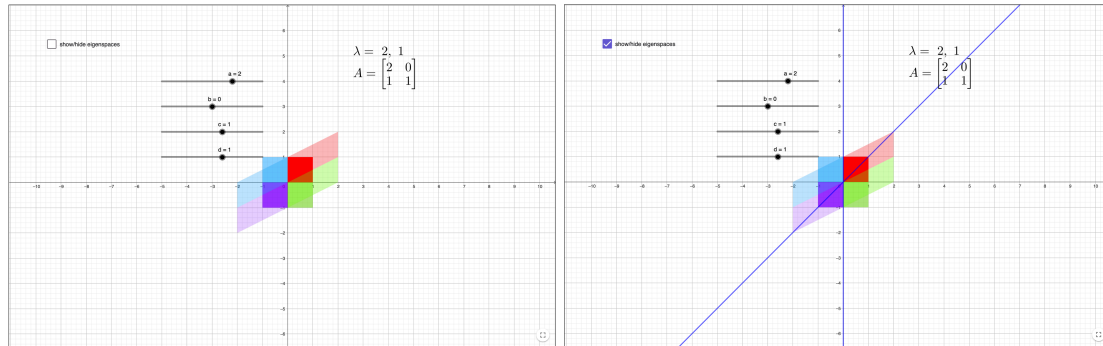
- the definition of eigenvectors and eigenvalues;
- the method for calculating eigenvalues from the characteristic polynomial and calculating eigenvectors from the relevant homogeneous equation;
- the relationship between eigenvalues and the trace and determinant;
- diagonalisation and the relationship between the geometric and algebraic multiplicity of eigenvalues;
- symmetric matrices and results about orthogonal diagonalisation.

We can examine these topics in various ways using the app presented here. Let us give some examples to demonstrate.

3.1. On the interpretation of the definition of eigenvalues and eigenvectors.

The definition of an eigenvector/eigenvalue is quite innocuous. However, its importance in both the theory and in applications of linear algebra cannot be overstated. One interpretation of the definition is that when searching for eigenvectors we are looking for vectors that remain on the same line through the origin. This of course means that they are only scaled and the scale factor is the corresponding eigenvalue. An immediate consequence of linearity is that *any* vector on that line will also remain on that line, so we can spot eigenvectors by looking for lines that are invariant under the action of the matrix. In fact, although this is not usually encountered in a first course on linear algebra, searching for invariant subspaces of a linear transformation is in fact one of the key tools in understanding them and so viewing things this way already introduces a fundamental point of view that becomes important later on.

In the app we can see these invariant subspaces explicitly in the blue lines and when this is introduced to students for the first time it can be pointed out that these can be identified by looking at where the blue line intersects the edge of the dark square and where it intersects the edge of the lighter parallelogram. These two intersection



(A) Transformation with eigenspaces suppressed

(B) The solution to the problem

FIGURE 2. An exercise for students to determine the eigenspaces of the transformation shown

points “match” in the sense that they are the same distance (proportionally) along the edge of the shapes. An easy first example is the one given in Figure 1, in which the eigenspace passes through the corners (in general they are much easier to spot when they pass through the corners of the coloured boxes; either along the axes or along the lines $y = \pm x$).

Students can be asked to identify the eigenvalue corresponding to each eigenspace. The eigenvalues are shown in the top right but which value corresponds with which space is not displayed.

After students have been shown the eigenspaces for a few different matrices, they can then be asked to identify these spaces for some examples by turning off the “show eigenspaces” button in the top left. Again, examples where the eigenvectors are along the corners are easiest to identify. See Figure 2 for an example.

3.2. Multiplicity and diagonalisability. One of the most important results concerning eigenvectors that students will be exposed to in their first course is the theorem that states that the dimension of an eigenspace is never larger than the multiplicity of the corresponding eigenvalue. The case where any eigenvalue has a strict inequality here shows the failure of some matrices to be diagonalisable.

In order to visualise how this can happen using the app we can use the family of matrices

$$\left\{ \begin{bmatrix} 1 & 1 \\ 0 & x \end{bmatrix} \mid x \in \mathbb{R} \right\}.$$

For example, starting with the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ we can drag the slider for the bottom right entry down towards 1 and see the second eigenspace (which starts out in the direction $(1, 1)$) approach the x -axis. See Figure 3. The eigenvalues are also shown to be 1, 1 indicating the multiplicity is two. The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is in Jordan canonical form and not diagonalisable and it is a good exercise for students to then perform the calculations

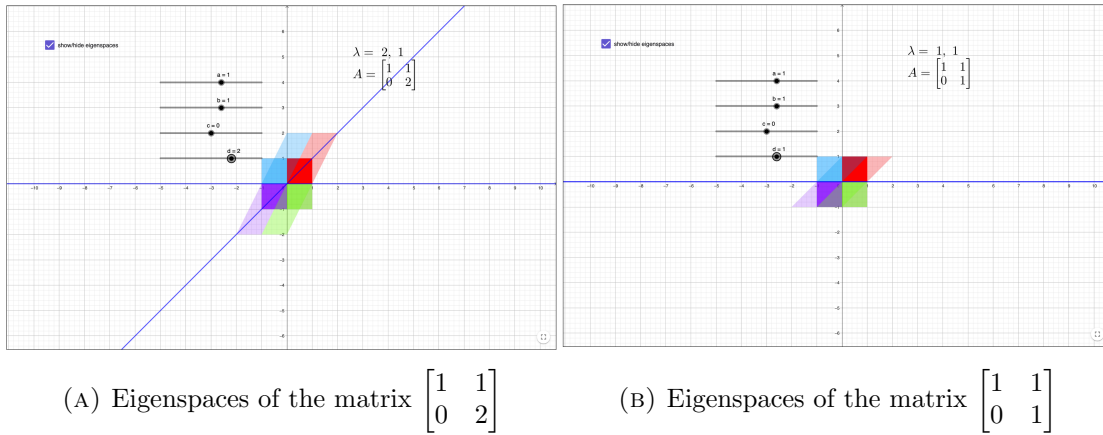


FIGURE 3. An example that demonstrates that the geometric multiplicity can be strictly less than the algebraic multiplicity

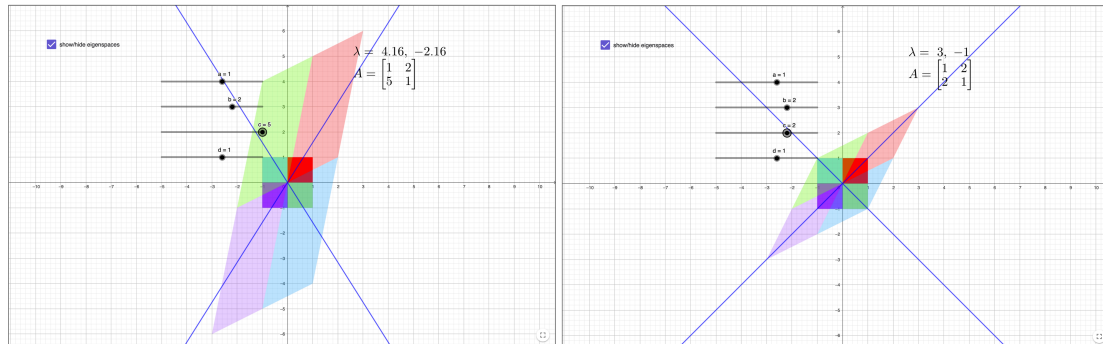
to determine the eigenvalues and eigenvectors of this matrix to verify what they have just seen.

Incidentally, by sliding the bottom right entry further until it reaches zero other important and interesting facts from elementary linear algebra can be observed explicitly. For example, when the slider reaches zero the parallelogram disappears. This is of course because the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ has rank one and the image of the square is therefore a line (which the app cannot display). The students can note that the eigenspaces however, remain, and one of these is the nullspace of the matrix (with eigenvalue zero).

3.3. Symmetric matrices. Another important result from undergraduate linear algebra concerns the eigenvalues and eigenvectors of symmetric matrices. Eigenvectors corresponding to distinct eigenvalues are always linearly independent but for real symmetric matrices they are in fact orthogonal. This can be observed using the app by taking for example the matrix $\begin{bmatrix} 1 & 2 \\ 5 & 1 \end{bmatrix}$ and sliding the bottom left entry towards 2. As this number approaches 2 we can see the angle between the eigenspaces approach $\frac{\pi}{2}$. See Figure 4. Students can experiment with different symmetric matrices to spot the pattern as a way of introducing this result.

3.4. Complex eigenvalues. The final example that we list here is the demonstration that eigenvalues for real matrices need not be real numbers. Depending on their background students may or may not have seen the Fundamental Theorem of Algebra before studying a first linear algebra course and therefore they may or may not realise that polynomials with real coefficients can have complex roots.

The examples we have given so far all have real eigenvalues. However, when we consider the action of 2×2 matrices on the coloured square, any matrix that rotates the square will not have any fixed lines and therefore no real eigenspaces or eigenvalues.



(A) Eigenspaces of the matrix $\begin{bmatrix} 1 & 2 \\ 5 & 1 \end{bmatrix}$ (B) Eigenspaces of the symmetric matrix $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

FIGURE 4. An example that demonstrates that symmetric matrices have orthogonal eigenspaces

An obvious example is the matrix

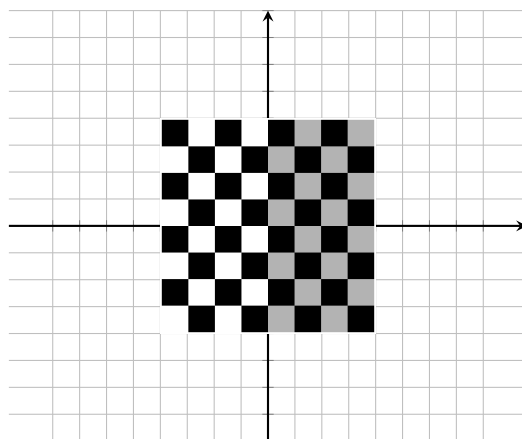
$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

which is a rotation by $\frac{\pi}{4}$. This matrix cannot be displayed in the app but the matrix $\begin{bmatrix} 0.7 & -0.7 \\ 0.7 & 0.7 \end{bmatrix}$ is a good replacement.

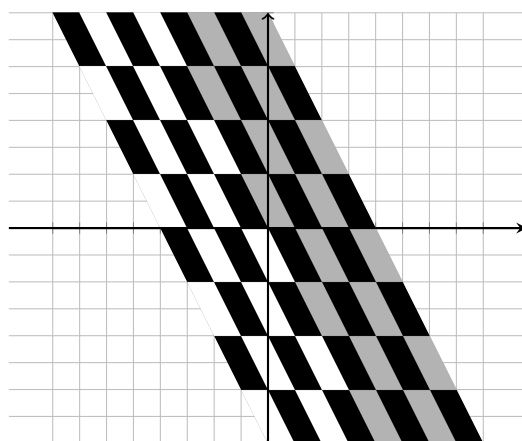
4. COMPLEMENTARY MATERIAL AND ACTIVITIES

The activities and explanations above can be complemented with various other tasks. A simple method for increasing engagement in large classes (which first year algebra courses tend to be) is by asking quiz questions that students can either discuss and/or respond to using their devices. An example of this is the Peer Instruction method, described in the context of linear algebra in [7]. We present below a couple of examples of questions along the lines of the discussion above that the author has used in linear algebra classes.

Example. Consider the square below, whose corners are at $(\pm 1, \pm 1)$.



The picture below shows the image of this square under a linear transformation $\mathbb{R}^2 \rightarrow \mathbb{R}^2$.



- (a) One of the eigenvectors of this transformation is $(1, 0)$. What is its corresponding eigenvalue?
- (b) Determine another eigenvector, that is not a multiple of $(1, 0)$, and give its corresponding eigenvalue.

5. CONCLUSION

We have described here a novel method of explaining the geometric meaning behind eigenvectors and eigenvalues and provided an interactive tool for students to use to investigate the various properties and results concerning these to which they might be exposed in a first linear algebra course. We advocate for such a tool to be used more widely in undergraduate courses on the grounds that it has the capacity to promote a deeper understanding of what is possibly the most important topic covered in a typical linear algebra course.

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