Moduli of special Lagrangian and coassociative submanifolds

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Introduction

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- they are minimal submanifolds
- nice moduli space of deformations
- the SYZ conjecture: "mirror symmetry is T-duality" nice geometric picture of mirror symmetry X,$\tilde{X}$ are dual special Lagrangian fibrations over the same base
- Dualities in M-theory: SYZ-like conjectures involving coassociative fibrations of G_2-manifolds.

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Minimal submanifolds

Let (X, g) be a Riemannian manifold

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Definition

A compact oriented submanifold $S \to X$ is called a *minimal submanifold* if it is a stationary point for the volume functional

$$\mathrm{vol}(S) = \int_S dvol_S$$

A classical result

Theorem

Let X be Kähler. Any compact complex submanifold of X is minimal

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Ex: A non-singular algebraic variety in $\mathbb{CP}^n$ is a minimal submanifold

Harvey and Lawson found a generalization of this result using calibrations

Let (X, g) be a Riemannian manifold.

Calibrations

Let (X, g) be a Riemannian manifold.

Definition

A *calibration* ϕ on X is a p-form such that

- ϕ is closed: $d\phi = 0$,
- for any $x \in X$ and oriented p-dimensional subspace $V \subseteq T_x X$, we have $\phi|_V = \lambda dvol$, where $\lambda \leq 1$ and $dvol$ is the volume form on V with respect to g.

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Subspaces such that $\lambda = 1$ are called *calibrated subspaces*.
An oriented p-submanifold $S \subseteq X$ is a *calibrated submanifold* of X with respect to ϕ if the tangent spaces of S are calibrated subspaces: $\phi|_S = dvol_S$.

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Let S be a compact calibrated submanifold. S has minimal volume amongst all submanifolds representing the same homology class.

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Proof.

Let S' be a compact submanifold with $[S] = [S']$. Then since ϕ is closed

$$\mathrm{vol}(S) = \int_S \phi = \int_{S'} \phi \leq \int_{S'} dvol_{S'} = \mathrm{vol}(S').$$

□

Calibrations

Why is this useful?

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The minimal submanifold equation $f : S \to X$ is second order in f
The calibrated submanifold condition $f^*\phi = dvol_S$ is first order in f.

Some examples

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$\mathrm{Spin}(7)$-manifolds: Cayley submanifolds

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Special Lagrangian and coassociative have a nice deformation theory (unobstructed). Associative and Cayley do not (obstructed). (See: McLean)

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A *Calabi-Yau manifold* is a Riemannian manifold with holonomy in $\mathrm{SU}(n)$.

On a Kähler manifold X with trivial canonical bundle ($K = \bigwedge^{n,0} T^*X$), every Kähler class admits a unique Calabi-Yau metric.

$\mathrm{SU}(n)$-structures

Reduction of structure to $\mathrm{SU}(n)$ can be defined using only

- A non-degenerate 2-form ω
- A complex n-form $\Omega = \Omega_1 + i\Omega_2$ which is locally decomposable: $\Omega = \theta_1 \wedge \cdots \wedge \theta_n$

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such that:

- $\omega \wedge \Omega = 0$,
- $\Omega \wedge \overline{\Omega} = \omega^n$ or $i\omega^n$,
- $\omega(I\,,\ )$ is positive with respect to the almost complex structure determined by Ω.

Integrability

The $\mathrm{SU}(n)$-structure is torsion free if and only if $d\omega = 0$, $d\Omega = 0$.

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In this case there is a torsion free $\mathrm{SU}(n)$-connection ∇ such that $\nabla\omega = 0$, $\nabla\Omega = 0$.

Precisely the requirement for a Calabi-Yau manifold.

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Special Lagrangians

The form $\Omega_1 = \mathrm{Re}(\Omega)$ is a calibration.

Definition

A submanifold $L \to X$ of a Calabi-Yau manifold is *special Lagrangian* if it is a calibrated submanifold with respect to Ω_1.

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Note: can replace Ω by $e^{-i\theta}\Omega$ and Ω_1 by $\cos(\theta)\Omega_1 + \sin(\theta)\Omega_2$.

Equivalent condition: $\omega|_L = 0$, $\Omega_2|_L = 0$ (hence the "Lagrangian" part of the name)

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special Lagrangians are much more rigid - we will see that compact ones have a finite dimensional moduli space of deformations of dimension b^1.

special Lagrangian tori exist in $b^1(T^n) = n$ dimensional families - just the right number for a torus fibration

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First order deformations

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Theorem (McLean)

A normal vector field X represents a first order deformation through special Lagrangian submanifolds iff

$$i_X\omega|_L \text{ is a harmonic 1-form on } L.$$

Moduli space of deformations

No obstructions to extending a first order deformation to an actual family

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notation: for $X \in T_L\mathcal{M}$ let $\theta_X = i_X\omega|_L$ be the corresponding harmonic form

Moduli space metric

$\mathcal{M}$ has a natural metric:

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Let $X, Y \in T_L\mathcal{M}$

define $g_\mathcal{M}$ on $\mathcal{M}$:

$$g_\mathcal{M}(X, Y) = \int_L \theta_X \wedge \star\theta_Y$$

called the L^2 moduli space metric

Local moduli space structure

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Theorem

α is closed, so (locally) we have a function

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such that $\alpha = du$:

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These are natural local affine coordinates $u_1, \ldots, u_{b^1(L)}$ on $\mathcal{M}$

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By same reasoning we get local affine coordinates $v : \mathcal{M} \to H^{n-1}(L, \mathbb{R})$.

Local moduli space structure

We have two sets of affine coordinates:
$u : \mathcal{M} \to H^1(L, \mathbb{R})$ and $v : \mathcal{M} \to H^{n-1}(L, \mathbb{R})$.

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Why do this?

The space $V = H^1(L, \mathbb{R}) \oplus H^{n-1}(L, \mathbb{R})$ has an obvious (n, n)-metric $\langle \, , \, \rangle$ and symplectic structure w:

$$
\begin{aligned}
\langle (a, b), (c, d) \rangle &= \frac{1}{2} \int_L a \wedge d + b \wedge c \\
w((a, b), (c, d)) &= \int_L a \wedge d - b \wedge c.
\end{aligned}
$$

Local moduli space structure

Theorem (Hitchin)

$F : \mathcal{M} \to H^1(L, \mathbb{R}) \oplus H^{n-1}(L, \mathbb{R})$ sends $\mathcal{M}$ to a Lagrangian submanifold. Moreover the natural L^2 metric on $\mathcal{M}$ is the induced metric.

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Let $u_1, \ldots, u_m$ be coords for $H^1(L, \mathbb{R})$, $(m = b^1(L))$.
Let $v_i, \ldots, v_m$ be dual coordinates for $H^{n-1}(L, \mathbb{R})$.
The (n, n) metric has form $\sum_i du_i dv_i$.

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$\mathcal{M}$ is a Lagrangian submanifold, so locally there is a function ϕ such that $v_i = \frac{\partial \phi}{\partial u_i}$.

Therefore the L^2-metric looks like

$$g_{\mathcal{M}} = \sum_{i,j} \frac{\partial^2 \phi}{\partial u_i \partial u_j} du_i du_j$$

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Then some linear combination $c_1 W_1 + c_2 W_2$ vanishes on $\mathcal{M}$ if and only if ϕ obeys the Monge-Ampère equation:

$$\det(\mathrm{Hess}(\phi)) = \mathrm{const}$$

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However this does not hold for all moduli spaces.
More on this soon.

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Conversely:

Theorem (Bryant)

Let g be a metric on T^n such that every non-zero harmonic 1-form is non-vanishing. Then (T^n, g) appears as a fibre in a special Lagrangian fibration. This is a local result: the total space need not be compact or complete.

Monge-Ampère revisited

Let $\mathcal{M}$ be the moduli spaces of deformations of L. Consider the enlarged moduli space

$$\mathcal{M}^c = \mathcal{M} \times H^1(L, \mathbb{R}/\mathbb{Z})$$

special Lagrangians with flat $\mathrm{U}(1)$-connections on them.

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Theorem

$\mathcal{M}^c$ is Kähler. The fibres of $\mathcal{M}^c \to \mathcal{M}$ are Lagrangian.

Monge-Ampère revisited

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If we start with X a semi-flat fibration then $\mathcal{M}^c$ deserves to be called the mirror of X

THANK YOU