**IGA Lectures, January 2010**
**Professor Frank Kutzschebauch**
**GROUP ACTIONS IN COMPLEX ANALYTIC GEOMETRY**

Frank Kutzschebauch has been a full professor at the University of Berne in Switzerland since 2005, and he is currently the director of the mathematics institute there. He obtained the Master of Science degree from Leningrad State University in 1991 and his doctorate from the University of Bochum in 1994. He has held visiting positions at Brandeis University, the Research Institute for Mathematical Sciences in Kyoto, Japan, and the University of Miami. He is one of the leaders in the emerging area of complex elliptic geometry. Some highlights of his research are the equivariant Oka-Grauert principle, established in joint work with P. Heinzner, his work with S. Kaliman on the density property of complex manifolds—both of these appeared in *Inventiones mathematicae*—his construction with H. Derksen of non-linearisable holomorphic actions of Lie groups on affine space, and his solution with B. Ivarsson of Gromov's Vaserstein problem.

While in Adelaide, Professor Kutzschebauch will give a series of eight lectures on the topic of *Group Actions in Complex Analytic Geometry*, as well as a colloquium talk on the Vaserstein problem. He summarises the goals and contents of his lectures as follows.

The study of symmetries is an important subject in the exploration of nature. Highly symmetric objects tend to be optimal in certain ways and the symmetries of an object can say a lot about the object itself. In mathematics this point of view was introduced by Felix Klein in his Erlangen Program. It has turned out to be very influential for mathematical research. The notion of a symmetry or a transformation is formalised in the mathematical notion of a group. In particular the notion of a Lie group, relating the algebraic notion of a group and the analytic notion of smoothness and introduced by Sophus Lie, has proved to be very fruitful. In complex analysis the study of symmetries is most interesting in higher dimensions. Here one finds spaces with a very rich geometry, so the study of their symmetries is important.

In the lecture series we start with a brief general introduction to group actions and Lie groups and another brief general introduction to several complex variables, in particular to Stein manifolds. These are the natural objects that carry a very rich function theory. Then we focus on actions of complex reductive Lie groups and of real compact Lie groups on Stein manifolds and Stein spaces. We give an overview of important results in geometric invariant theory on Stein spaces, developed mainly by Dennis Snow and Peter Heinzner, the complex analytic counterpart of David Mumford's geometric invariant theory, starting with Hilbert's famous theorem on the finite generation of the ring of invariants of a linear action of a compact Lie group on a finite dimensional complex vector space. The important results here are the existence of a categorical quotient, which is a Stein space, and a slice theorem in the spirit of Luna.

Then we shift our attention to Stein manifolds with very big, infinite-dimensional automorphism groups, first of all to complex affine space $\mathbb{C}^n$ for $n \geq 2$. We give an overview of known results about the automorphism group of $\mathbb{C}^n$, focusing on its Lie subgroups. Here the question of linearisation has attracted attention both in the complex analytic and the complex algebraic categories.

We will give examples of non-linearisable actions of Lie groups on affine space both in the algebraic setting, discovered by G. Schwarz, and in the analytic setting, discovered by H. Derksen and the speaker, commenting on their completely different nature.

As preparation for constructing these examples, we need to explain the Andersen-Lempert theorem, which roughly speaking is a tool for constructing holomorphic automorphisms with prescribed local behaviour. This theorem leads to complicated embeddings of Stein manifolds into affine space, which are used to construct the examples of non-linearisable Lie group actions. Other applications of the Andersen-Lempert theorem of geometric interest will be sketched.

Next we move to Stein manifolds with the density property. This notion was introduced by D. Varolin, generalising a property of the group of automorphisms of affine space which leads to the above-mentioned theorem of Andersen and Lempert. More precisely, a manifold $X$ has the density property if the Lie algebra generated by globally integrable holomorphic vector fields on $X$ is dense in the Lie algebra of all holomorphic vector fields on $X$. After summarising some properties of these manifolds we will give examples.

For this we will explain a construction, called affine modification (or Kaliman modification, although it goes back to Zariski) which turns out to be an important method for producing interesting examples both in the affine algebraic and the complex analytic setting. If one modifies affine space in a particular way, the resulting manifold will have the density property: this is a result of Kaliman and the speaker.

Then we prove a criterion which ensures the density property, developed by Kaliman and the speaker, and explain how it can be used to prove that the complement of an algebraic subvariety of $\mathbb{C}^n$ of codimension at least 2 has the density property, and also that all linear algebraic groups except $\mathbb{C}$ and tori have the density property.

In the last lecture we explain a similar notion, the so-called volume density property, and present some basic facts about Stein manifolds possessing this property. These are very recent, unpublished results of Kaliman and the speaker.

The titles of the eight lectures are as follows.

1. Several complex variables, Stein manifolds
2. Lie groups and group actions
3. The idea of invariant theory, Hilbert's finiteness result
4. Geometric invariant theory on Stein spaces
5. Automorphisms of affine space
6. Non-linearisable group actions on affine space
7. Stein manifolds with the density property
8. Stein manifolds with the volume density property

The lectures will be given two at a time, as follows, in the board room of the School of Mathematical Sciences on level 3 of 10 Pulteney Street:

1:10 pm to 3:00 pm on Friday 8 January

10:10 am to 12:00 pm on Fridays 15, 22, and 29 January