An Overview of Mirror Symmetry and Symplectic Topology

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Outline

I) Symplectic Topology

- Symplectic Geometry
- Symplectic Topology
- Lagrangian manifolds
- Gromov-Floer theory

II) Mirror Symmetry

- Pseudo-historical overview
- Homological mirror symmetry (HMS)
- Geometric mirror symmetry (SYZ)
- Theme and variations

Symplectic Geometry

- **Definition:** A symplectic manifold (M, ω) is a smooth manifold equipped with a closed 2-form ω such that ω^n is a volume form.

- History: Celestial mechanics - time evolution of position
  - Lagrange (1808): Introduce variables (p_i, q_i), first order equation for evolution using the symplectic form $\omega = \sum_i dp_i \wedge dq_i$

    Notation: (q_i) are coordinates for a manifold Q

    $p_i \leftrightarrow$ "dq_i" $\qquad T^*Q$ cotangent
  - Hamilton (1838): Develop theory of mechanics
  - 19-20th century: Lagrange + Hamilton give a common language for physics (See Guillemin-Sternberg "Symplectic methods...")

Uspekhi Mat. Nauk 41:6 (1986), 3–18 Russian Math. Surveys 41:6 (1986), 1–21

First steps in symplectic topology(1)

V.I. Arnol'd

CONTENTS

Introduction

By symplectic topology I mean the discipline having the same relation to ordinary topology as the theory of Hamiltonian dynamical systems has to the general theory of dynamical systems. The correspondence here is similar to that between real and complex geometry.

A complex linear space can be considered as an even-dimensional real space, furnished with additional structure (the operation of multiplication by i). However, complexification of a theory does not boil down to reducing the pile of spaces and the addition of a new operation: all the concepts take on new meanings. For example, complex subspaces or operators are not the same as subspaces or operators in the underlying real space. Thus complex geometry is an analogue of real geometry, but not a particular case of it.

In precisely the same way, symplectic geometry can be considered, of course, as ordinary geometry in the presence of additional structure. But it

(1)The papers of V.I. Arnol'd, A.N. Dranishnikov, E.V. Shchepin, and V.V. Fedorchuk brought together in this issue are reports of plenary sessions held on 27–29 May 1986 at the "Aleksandrov Colloquium" (jointly with a session of the Moscow Mathematical Society and the All-Moscow topological seminar in the name of P.S. Aleksandrov (the topological association)), dedicated to the 90th anniversary of Aleksandrov's birth (*Editor's note*).

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is possible to adopt another point of view in which symplectic geometry can be considered rather as an analogue of ordinary geometry in its own right. For example, the symplectic group may be considered not as a subgroup of a group of matrices of even order, but as the simple Lie group C_k, having equal rights with the group of non-singular matrices A_k, not least in having a distinctive system of roots and so on.

Questions of symplectic topology, which we shall be speaking about later, can be considered as questions of ordinary topology in the presence of additional structure. But of much greater interest to me is not the use of ordinary topology in the study of objects of symplectic geometry, but divining symplectic results by means of "symplectization".

Symplectization transforms not only the initial objects (manifolds, maps, ...), but also the whole theory. For example, the concepts of boundary and homology theory in symplectic topology are quite different from the ordinary ones. The dimension of a "symplectic boundary" should not be one, but two units less than the dimension of the original manifold (lowering dimension in symplectic geometry is always accomplished in two stages, one of which is section and the other projection).

I do not intend here to formalize these nebulous ideas(1), but pass to specific conjectures which they give rise to (omitting rather lengthy intermediate considerations).

Some of the conjectures of this type published in the years 1965–1976 ([9]–[12]) have recently been proved by Conley, Zehnder, Sicorav, Gromov, and others, and powerful new techniques have been developed. It seems to me that now is the time to return to other conjectures of this type and even perhaps to look at the whole programme of symplectization.

Odd-dimensional variants (related to contact topology) are also considered below.

The author thanks A.V. Alekseev, M.L. Byaly, Yu.V. Chekanov, Ya.M. Eliashberg, D.B. Fuks, V.L. Ginzburg, A.V. Givental', V.P. Kolokol'tsov, V.V. Kozlov, V.P. Maslov, S.P. Novikov, J. Nye, L.V. Polterovich, E.V. Shchepin, A.I. Shnirel'man, and V.A. Vasil'ev for numerous useful discussions.

§1. Is there such a thing as symplectic topology?

A *symplectic structure* on a manifold is a closed non-degenerate 2-form. The simplest example of a symplectic manifold is the plane; the (oriented) element of area provides the symplectic structure.

A *symplectic diffeomorphism* (or *symplectomorphism*) is a diffeomorphism preserving the symplectic structure. It is clear that this condition puts a

(1)As regards symplectic boundaries see the theory of Lagrangian cobordism in [1]–[7]; the complexification of the concept of boundary is a branching divisor, $\mathbf{Z}_2$ is replaced by $\mathbf{Z}$, Stiefel–Whitney classes by Chern classes, and so on (see [8]).

Rigidity vs Flexibility

Eliashberg & Gromov ~1980: Symplectic topology exists.
Proved that diffeomormiphsms preserving ω are C^0 closed

① Which manifolds admit symplectic forms? how many?

- In the closed case completely open.
- Open manifolds: Gromov $\Rightarrow$ Complete flexibility (h-principle)
- Manifolds with ∂: Eliashberg $\Rightarrow$ Rigidity for existence

Subtlety for uniqueness:
- Infinitely many different structures in dim ≥ 6 (McLean, A-Seidel)
- Eliashberg-Floer-McDuff Uniqueness for $\mathbb{R}^4$

Everything in the world is a Lagrangian manifold

Weinstein

Definition: A Lagrangian submanifold $L \subset M$ is an n-dimensional smooth manifold such that $\omega|L \equiv 0$.

② What is the space of Lagrangians?

- Gromov ~72: complete flexibility for immersions.
- Gromov ~83: S^3 does not embed in $\mathbb{R}^6$

Arnol'd Conjecture: Every closed exact Lagrangian in a cotangent bundle is isotopic to the zero section.

Thm: Every such Lagrangian is homotopy equivalent to the zero section

Gromov-Floer Theory

Definition: An almost complex structure J on M is compatible with the symplectic form if
$\omega(\cdot, J\cdot)$ is a metric

Space of cpatible J is contractible

Gromov '83: Can count maps $u:(\Sigma, j) \longrightarrow M$ satisfying
(Riemann surface)
$du \circ j = J \circ du$ pseudo-holomorphic

Floer '86: Generalize Morse theory: Replace gradient flow by count of cylinders

Floer Cohomology $\begin{cases} \text{- Symplectic manifolds} \\ \text{- Lagrangian pairs} \\ \text{- } \dots \end{cases}$

Lagrangian Floer Cohomology

Idealised situation: $L_1, L_2 \subset M$ Lagrangians $\rightsquigarrow$ $\underline{HF^*(L_1, L_2)}$

<u>Properties:</u>

- $HF^*(L_1, L_2) \cong HF^*(L_1, \phi(L_2))$ if ϕ is isotopic to identity & Hamiltonian
- $H^*(L) \Rightarrow HF^*(L, L)$ spectral sequence — differentials count $\underline{\text{discs}}$

Used by Floer to understand $Q \cap \phi(Q)$ in T^*Q.

<u>Question:</u> What can we say about $HF^*(L_1, L_2)$ when L_1, L_2 not explicitly given?

Donaldson-Fukaya: $HF^*(L_1, L_2)$ are morphism spaces in a category.

<u>e.g.</u>: $HF^*(L_1, L_1)$ is an algebra, with $HF^*(L_1, L_2)$ a module

Origins of Mirror Symmetry

Witten '88: Interpret count of J-holomorphic curves as "correlation functions" of topological field theory called the A-model twist of standard σ-model on a Calabi-Yau manifold X. Related to symplectic geometry

There is another twist called the B-model = complex geometry.

Mirror Symmetry: Equivalence of A-model to B-model

Candelas-de la Ossa-Green-Parkes '91: Consider the Quintic 3-fold

$$Q = \{x_1^5 + \cdots + x_5^5 = 0\} \subset \mathbb{P}^4$$

Count of rational curves of a given degree is equal to period integrals on the mirror.

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TABLE 4
The numbers of rational curves of degree k for $1 \leqslant k \leqslant 10$

k	n_k
1	2875
2	6 09250
3	3172 06375
4	24 24675 30000
5	22930 58888 87625
6	248 24974 21180 22000
7	2 95091 05057 08456 59250
8	3756 32160 93747 66035 50000
9	50 38405 10416 98524 36451 06250
10	70428 81649 78454 68611 34882 49750

number of conics [28] (rational curves of degree two). Clemens has shown [30] that $n_k \neq 0$ for infinitely many k and has conjectured that $n_k \neq 0$ for all k, but it seems that the direct calculation of these numbers becomes difficult beyond $k = 2$ (see also ref. [28]). It is however straightforward to develop the series (5.12) to more terms and to find the n_k by comparison with (5.13). We present the first few n_k in table 4. These numbers provide compelling evidence that our assumption about the form of the prefactor is in fact correct. The evidence is not so much that we obtain in this way the correct values for n_1 and n_2, but rather that the coefficients in eq. (5.12) have remarkable divisibility properties. For example asserting that the second coefficient 4,876,875 is of the form $2^3 n_2 + n_1$ requires that the result of subtracting n_1 from the coefficient yields an integer that is divisible by 2^3.

3 Branches of Mirror Symmetry

① Are these predictions correct?

- Physics predictions for CY complete intersections.
  Extended to positive anticanonical bundle
- Genus 0 proved by Givental '96.

② What phenomena are detected by mirror symmetry?

Equivalence of categories between
- Coherent sheaves
- Fukaya category

Kontsevich '94

③ Why are there mirror manifolds?

Existence of dual torus fibrations

Strominger, Yau, Zaslow '96

What Mirror Symmetry doesn't Know

B-side: ① Conjecturally, as coarse as "crepant birational geometry"

② In practice, all results near "large complex structure limit"

A-side: ① Smooth structures

② Quantitative information

Calabi-Yau

$c_1(X)=0$

Cheme

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Conjecture: X proper, smooth, complex manifold with Kähler form ω. and holomorphic volume form Ω. $\exists\ X^\vee$ such that

① $\mathcal{F}(X) \cong D^b\,Coh(X^\vee)$ and $\mathcal{F}(X^\vee) \cong D^b\,Coh(X)$ HMS

② $\exists$ dual singular Special Lagrangian fibrations SYZ

$$\begin{array}{ccccc} T^n \to X^\vee & & & & X \leftarrow T^n \\ & \searrow & & \swarrow & \\ & & B & & \end{array}$$

minimal volume

$B \hookleftarrow$ manifold of $\dim_{\mathbb{R}} n$

(1+2) or (2) $\Rightarrow$ $H^{p,q}(X) \cong H^{n-p,q}(X^\vee)$

Counter example: Rigid Calabi-Yaus

Known Results

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<table>
<tr><th></th><th colspan="2">HMS</th><th>SYZ</th></tr>
<tr><td>Elliptic curves.</td><td colspan="2">Polischuck & Zaslow '98
Polischuck '02</td><td rowspan="2">Trivial</td></tr>
<tr><td>Abelian varieties</td><td>Fukaya '03</td><td></td></tr>
<tr><td>K3</td><td>Seidel '04
(quartic)</td><td colspan="2">Kontsevich-Soibelman '01</td></tr>
<tr><td>CY hypersurface</td><td colspan="2"></td><td>Gross-Siebert '03-08</td></tr>
<tr><td>Products</td><td colspan="2">A-Smith '09</td><td></td></tr>
</table>

↓ Coh(X) understood not over $\mathbb{C}$ by over power series ring

Variation 1: Fano varieties

Assume now that X (smooth & proper) admits a section of the anticanonical bundle (ample anticanonical)

Conjecture: The mirror of X is a Landau-Ginzburg potential $W: Y \to \mathbb{C}$. open Calabi Yau

holomorphic

(1a) $D^b Coh(X) \cong \mathcal{F}(W)$

(1b) $\mathcal{F}(X) \cong D^b Sing(W)$

HMS Kontsevich '96

(2) If D = anticanonical divisor, then $\exists$ dual fibrations

$$\begin{array}{ccccc} X - D & & & & Y \\ & \searrow & & \swarrow & \\ & & B & & \end{array}$$

open CY

Fukaya, Auroux

Why $D^b\mathrm{Sing}$?

Kontsevich '98: W determines a deformation of $D^b\mathrm{Coh}(Y)$

Orlov '01: Objects go to zero unless their support intersects $\mathrm{Crit}(W)$

$\Rightarrow$ Get a spectral sequence

$$\mathrm{Hom}_{D^b\mathrm{Coh}(Y)}(\mathcal{S},\mathcal{S}) \Rightarrow \mathrm{Hom}_{D^b\mathrm{Sing}(W)}(\mathcal{S},\mathcal{S})$$

Mirror to the spectral sequence

$$H^*(L) \Rightarrow HF^*(L,L)$$

$H^*(L) \sim HF^*(L,L)$ in $\underline{X \setminus D}$

Slogan: W counts holomorphic discs!

Known Results

	$D^b Coh(X)$	$\mathcal{F}(X)$	SYZ
del Pezzo surfaces.	Auroux-Katzarkov-Orlov '04		
Toric varieties	A '07	Fukaya-Oh-Ohta-Ono '08-'10	Fukaya-Oh-Ohta-Ono '08-'10
Complete intersections			Gross-Siebert 01'-'10

Variation 2: Open Calabi-Yau

$X \setminus D$ smooth with holomorphic volume form (simple poles at infinity)

Conjecture: $\exists\, Y$ such that

(HMS)

① $D^b Coh_c(X \setminus D) \simeq \mathcal{F}(Y)$ (Backwards as well)

①' $D^b Coh(X \setminus D) \simeq \mathcal{W}(Y)$

②
$$\begin{array}{ccc} X\setminus D & & Y \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$
dual fibrations

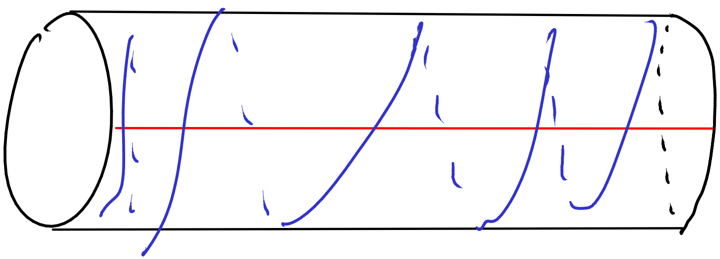
Remark: Work over $\mathbb{C}$ on A-side

Why Wrapping?

- Existence of s.s implies $HF^\vee(L_1, L_2)$ always finite rank.
- Since $X \setminus D$ is not proper, $\mathrm{Ext}^\vee(F_1, F_2)$ likely infinite

<u>e.g:</u> $\mathbb{A}^1 \setminus \{0\}$ $\qquad \Gamma(\mathcal{O}_{\mathbb{A}^1 \setminus \{0\}}) =$ Lauren poly

$$= \mathbb{C}[t, t^{-1}]$$



Known Results

	$\mathcal{F}(X)$	$\mathcal{W}(X)$	SYZ
$(\mathbb{C}^*)^n$	Fukaya-Seidel-Smith '07	FSS '07 A '10	
$\mathbb{C}^2 \setminus$ conic	Pascaleff '10		Auroux '08
X del Pezzo Tot $\mathcal{O}_{-K}(X)$	Seidel '09		

Mirror symetry for $\mathbb{C}^n$ - - - -

Variation 3: Smooth Varieties

Katzarkov '02: All smooth varieties X should have Landau-Ginzburg mirrors.

? So far, no situation where mirror of $\mathrm{Coh}(X)$ is understood

<table>
<tr><th></th><th>$\mathcal{F}(X)$</th><th>SYZ</th></tr>
<tr><td>Closed Curves</td><td>Seidel '08, Efimov '09</td><td rowspan="4">A - Auroux Katzarkov '10</td></tr>
<tr><td>Hypersurfaces in $\mathbb{CP}^n$</td><td></td></tr>
<tr><td>$\mathbb{CP}^1 \setminus \{\text{points}\}$</td><td>A - Auroux - Efimov - Katzarkov - Orlov '10</td></tr>
<tr><td>$\mathbb{CP}^n \setminus \{\text{hyperplanes}\}$</td><td>Sheridan '10</td></tr>
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