On the holonomy fibration

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based on work with Alejandro Cabrera and Marco Gualtieri

Workshop on
Geometric Quantization
Adelaide, July 2015

Introduction

General theme:

Hamiltonian LG-spaces $\leftrightarrow$ q-Hamiltonian G-spaces

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow[\Psi]{} & L\mathfrak{g}^* \\ \big\downarrow{\scriptstyle /L_0G} & & \big\downarrow{\scriptstyle /L_0G} \\ M & \xrightarrow[\Phi]{} & G \end{array}$$

Many aspects of this correspondence are not fully understood.

Introduction

	Ham. LG-spaces	q-Ham. G-spaces
Volume forms	?	Y
Equivariant cohomology	(?)	Y
Kirwan surjectivity	Y	(Y)
Norm-square localization	Y	(?)
Quantization	(?)	Y
Kähler structures	(Y)	?
‘Poisson’ structures	?	(Y)

Holonomy fibration

$$\begin{array}{c} \mathcal{A} = \Omega^1(S^1, \mathfrak{g}) \ \circlearrowleft LG \\ \Big\downarrow \text{Hol} \ /L_0 G \\ G \end{array}$$

$$g.A = \mathrm{Ad}_g(A) - \partial g\ g^{-1}.$$

Suppose $\mathfrak{g}$ has invariant metric, denoted "$\cdot$". $\rightsquigarrow$ central extension:

$$\widehat{L\mathfrak{g}} = \mathbb{R} \oplus L\mathfrak{g}$$

with bracket

$$[t_1 + \xi_1, t_2 + \xi_2]_{\widehat{L\mathfrak{g}}} = \int_{S^1} \xi_1 \cdot \partial \xi_2 + [\xi_1, \xi_2]_{L\mathfrak{g}}.$$

Have LG-equivariant isomorphism

$$\mathcal{A} = \{1\} \times L\mathfrak{g}^* \subset \widehat{L\mathfrak{g}}^*.$$

Holonomy fibration

Basic facts from Poisson geometry

- Let $\mathfrak{k} = \mathrm{Lie}(K)$. Then $\mathfrak{k}^*$ has a K-invariant Poisson structure. (Leaves = coadjoint orbits.)
- For central extension

$$0 \to \mathbb{R} \to \widehat{\mathfrak{k}} \to \mathfrak{k} \to 0$$

  the level sets $\{\lambda\} \times \mathfrak{k}^* \subset \widehat{\mathfrak{k}}^*$ are K-equivariant Poisson submanifolds.
- If M is Poisson, $H \circlearrowright M$ a principal action by Poisson automorphisms, then M/H is Poisson.

Holonomy fibration

We'd like to apply this to our setting:

$$0 \to \mathbb{R} \to \widehat{L\mathfrak{g}} \to L\mathfrak{g} \to 0$$

with LG acting on

$$\mathcal{A} = \{1\} \times L\mathfrak{g}^* \subset \widehat{L\mathfrak{g}^*}$$

(symplectic leaves = coadjoint LG-orbits) and

$$\mathcal{A}/L_0G \cong G.$$

But: G does not have a reasonable $LG/L_0G \cong G$-equivariant Poisson structure.

Problem: $\dim \mathcal{A} = \infty$, $\dim LG = \infty$.

It's not even obvious what we mean by 'Poisson structure' on $\mathcal{A}$.

- Bivector field $\pi \in \Gamma(\wedge^2 T\mathcal{A})$? Infinite rank??
- Bilinear forms $\{\cdot,\cdot\}$ on smooth functions? Domain??

We'll show: The 'Lie-Poisson structure' on $\mathcal{A}$ makes sense as a *Dirac structure*, and descends to a *Dirac structure* on G.

Dirac geometry

Dirac geometry was introduced by T. Courant and A. Weinstein in 1989.

- $\mathbb{T}M = TM \oplus T^*M,.$
- $\langle v_1 + \alpha_1, v_2 + \alpha_2 \rangle = \alpha_1(v_2) + \alpha_2(v_1),$
- $[\![v_1 + \alpha_1, v_2 + \alpha_2]\!] = [v_1, v_2] + \mathcal{L}_{v_1}\alpha_2 - \iota_{v_2}\mathrm{d}\alpha_1 + \iota_{v_1}\iota_{v_2}\eta.$

where $\eta \in \Omega^3_{cl}(M).$

Definition

$E \subset \mathbb{T}M$ is a Dirac structure if

- $E = E^\perp,$
- $\Gamma(E)$ closed under $[\![\cdot,\cdot]\!]$.

Dirac geometry

Examples

1. $\omega \in \Omega^2(M) \rightsquigarrow \mathrm{Graph}(\omega)$ is a Dirac structure $\Leftrightarrow \mathrm{d}\omega = 0$.
2. $\pi \in \Gamma(\wedge^2 TM) \rightsquigarrow \mathrm{Graph}(\pi)$ is a Dirac structure $\Leftrightarrow$ π is Poisson.
3. Conversely, a Dirac structure $E \subseteq \mathbb{T}M$ is a Poisson structure $\Leftrightarrow E \cap TM = 0$.
4. Lie-Poisson structure: $E \subseteq \mathbb{T}\mathfrak{g}^*$ spanned by sections
$$e(\xi) = \xi_{\mathfrak{g}^*} + \langle \mathrm{d}\mu, \xi \rangle, \quad \xi \in \mathfrak{g}.$$
5. Cartan-Dirac structure: $E \subseteq \mathbb{T}G$ spanned by sections
$$e(\xi) = \xi_G + \tfrac{1}{2}(\theta^L + \theta^R) \cdot \xi, \quad \xi \in \mathfrak{g}$$
is a Dirac structure wrt $\eta = \frac{1}{12}\theta^L \cdot [\theta^L, \theta^L] \in \Omega^3(G)$. Here $\theta^L = g^{-1}\mathrm{d}g,\ \theta^R = \mathrm{d}g\ g^{-1}$.

Dirac geometry

The definitions also work for Hilbert manifolds M.

Definition

A Dirac structure $E \subseteq \mathbb{T}M$ is called a

- Poisson structure $\Leftrightarrow \quad E \oplus TM = \mathbb{T}M$,
- weak Poisson structure $\Leftrightarrow \quad E \cap TM = 0$.

The leaves of a weak Poisson structure are weakly symplectic.

Dirac geometry

Example

- $\mathcal{A} = \Omega^1_{H^r}(S^1, \mathfrak{g})$ connections of Sobolev class $r \geq 0$
- $LG = \mathrm{Map}_{H_{r+1}}(S^1, G)$ loop group
- Dirac structure $E \subseteq \mathbb{T}\mathcal{A}$ spanned by

$$e(\xi) = \xi_{\mathcal{A}} + \langle \mathrm{d}A, \xi \rangle, \quad \xi \in L\mathfrak{g}$$

where

$$\xi_{\mathcal{A}}|_A = \partial\xi + [A, \xi].$$

Then E is a weak Poisson structure: $E \cap T\mathcal{A} = 0$. But $\mathbb{T}\mathcal{A} \neq E \oplus T\mathcal{A}$. Note

$$E|_A = \mathrm{graph}(\partial_A), \quad \partial_A \colon \Omega^0_{H_{r+1}}(S^1, \mathfrak{g}) \to \Omega^1_{H_r}(S^1, \mathfrak{g})$$

skew-adjoint operator $\partial_A = \partial + \mathrm{ad}_A$ with dense domain.

Reduction of Dirac structures

Suppose $H \circlearrowright M$ preserves $\eta \in \Omega^3_{cl}(M)$. Then $H \circlearrowright \mathbb{T}M$ by automorphisms of $\langle \cdot, \cdot \rangle$, $[\![\cdot, \cdot]\!]$.

Definition

$\varrho \colon \mathfrak{h} \to \Gamma(\mathbb{T}M)$ defines generators for the action if

$$[\![\varrho(\xi), \cdot]\!] = \frac{d}{dt}\Big|_{t=0} \exp(-t\xi)^*$$

on $\Gamma(\mathbb{T}M)$.

Examples

$G \circlearrowright \mathbb{T}\mathfrak{g}^*$, $G \circlearrowright \mathbb{T}G$, $LG \circlearrowright \mathbb{T}\mathcal{A}$ have generators $\varrho(\xi) = e(\xi)$.

$$e(\xi) = \xi_{\mathfrak{g}^*} + \langle \mathrm{d}\mu, \xi \rangle, \quad \xi \in \mathfrak{g}$$
$$e(\xi) = \xi_G + \tfrac{1}{2}(\theta^L + \theta^R) \cdot \xi, \quad \xi \in \mathfrak{g}$$
$$e(\xi) = \xi_{\mathcal{A}} + \langle \mathrm{d}A, \xi \rangle, \quad \xi \in L\mathfrak{g}$$

Reduction of Dirac structures

Suppose $H \circlearrowright M$ is a principal action preserving η, and that

$$\varrho\colon M \times \mathfrak{h} \to \mathbb{T}M$$

defines generators for $H \circlearrowright \mathbb{T}M$.

Theorem (Bursztyn-Cavalcanti-Gualtieri)

Suppose $J = \varrho(M \times \mathfrak{h}) \subseteq \mathbb{T}M$ is isotropic. Then $\langle \cdot, \cdot \rangle$, $[\![\cdot, \cdot]\!]$ descend to

$$\mathbb{T}M /\!\!/ H = (J^\perp / J)/H.$$

Furthermore, if $E \subset \mathbb{T}M$ is an H-invariant Dirac structure with $J \subseteq E$, then

$$E /\!\!/ H = (E/J)/H$$

is a Dirac structure.

One has $\mathbb{T}M /\!\!/ H \cong \mathbb{T}(M/H)$ but this depends on a choice..

Reduction of Dirac structures

Have an exact sequence

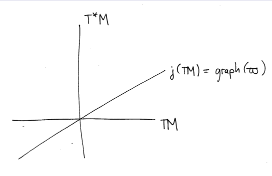
$$0 \to T^*(M/H) \to \mathbb{T}M /\!\!/ H \to T(M/H) \to 0.$$

To identify $\mathbb{T}M /\!\!/ H \cong \mathbb{T}(M/H)$ one needs an isotropic splitting

$$T(M/H) \to (\mathbb{T}M) /\!\!/ H.$$

Equivalently, need H-equivariant isotropic splitting $j\colon TM \to \mathbb{T}M$ with $J \subseteq j(TM)$.

Such a splitting is described by a 2-form ϖ, with $\mathrm{d}\varpi = p^*\eta$.



Reduction of Dirac structures

Given $H \circlearrowright M$ and $\varrho\colon M \times \mathfrak{h} \to \mathbb{T}M$ as above, we have:

Any connection 1-form $\theta \in \Omega^1(M, \mathfrak{h})^H$ determines an isotropic splitting.

Explicitly, writing $\varrho(\xi) = \xi_M + \alpha(\xi)$, and letting

$$c(\xi_1, \xi_2) = \iota((\xi_1)_M)\alpha(\xi_2),$$

we have

$$\varpi = -\alpha(\theta) + \tfrac{1}{2}c(\theta, \theta).$$

Reduction of the Lie-Poisson structure on $\mathcal{A}$

Back to our setting:

$$LG \circlearrowright E \subseteq \mathbb{T}\mathcal{A}, \qquad \varrho\colon L\mathfrak{g} \to \Gamma(\mathbb{T}\mathcal{A}), \qquad \mathcal{A}/L_0G = G.$$

Hol: $\mathcal{A} \to G$ has an 'almost canonical' connection $\theta \in \Omega^1(\mathcal{A}, L_0\mathfrak{g})$, depending on choice of a 'bump function' on $[0, 1]$. Constructed by caloron correspondence of Michael Murray and Raymond Vozzo.

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & L_0Q \\ \downarrow{\scriptstyle /L_0G} & & \downarrow{\scriptstyle /L_0G} \\ G & \longrightarrow & L_0(G \times S^1) \end{array}$$

with the G-bundle $Q = (G \times \mathbb{R} \times G)/\mathbb{Z} \to G \times S^1$.

Reduction of the Lie-Poisson structure on $\mathcal{A}$

For any choice of bump function, get connection θ and hence $\varpi \in \Omega^2(\mathcal{A})^{LG}$ as above.

- ϖ independent of choice of bump function.
- $d\varpi = \mathrm{Hol}^* \eta$, with $\eta \in \Omega^3(G)$ the Cartan 3-form.

This 2-form also appeared in the '98 AMM paper:

$$\varpi = \frac{1}{2} \int_0^1 \mathrm{Hol}_s^* \theta^R \cdot \frac{\partial}{\partial s} \mathrm{Hol}_s^* \theta^R \in \Omega^2(\mathcal{A})$$

where $\mathrm{Hol}_s \colon \mathcal{A} \to G$ is the holonomy from 0 to s.

Reduction of the Lie-Poisson structure on $\mathcal{A}$

Theorem (Cabrera-Gualtieri-M)

The reduction of $(\mathbb{T}\mathcal{A}, E)$ by $L_0 G$, using the standard generators ϱ and the isotropic splitting defined by ϖ, is the Cartan-Dirac structure on G.

Reduction of the Lie-Poisson structure on $\mathcal{A}$

Remark

Can also consider twisted loop groups $\rightsquigarrow$ *twisted Cartan-Dirac structure.*

Remark

Similar discussion of S^1 *with* n *marked points* $\rightsquigarrow$ *interesting Dirac structure on* G^n. *(Cf. Li-Bland, Severa.)*

Reduction of morphisms

Let M_1, M_2 be manifolds with closed 3-forms η_1, η_2.

A map $\Phi\colon M_1 \to M_2$ together with a 2-form $\omega \in \Omega^2(M_1)$ defines a *relation* $(\mathbb{T}M_1)_{\eta_1} \dashrightarrow (\mathbb{T}M_2)_{\eta_2}$, where

$$v_1 + \alpha_1 \sim v_2 + \alpha_2 \Leftrightarrow v_2 = \Phi_* v_1, \quad \alpha_1 = \Phi^*\alpha_2 + \iota_{v_1}\omega.$$

Given Dirac structures $E_i \subset (\mathbb{T}M_1)_{\eta_i}$, we say that

$$(\Phi, \omega)\colon ((\mathbb{T}M_1)_{\eta_1}, E_1) \dashrightarrow ((\mathbb{T}M_2)_{\eta_2}, E_2)$$

is a Dirac morphism if

- $\Phi^*\eta_2 = \eta_1 + \mathrm{d}\omega$
- Every $x_2 \in (E_2)_{\Phi(m)}$ is (Φ, ω)-related to a unique element $x_1 \in (E_1)_m$.

Reduction of morphisms

Example

A Dirac morphism $(\Phi, \omega)\colon (\mathbb{T}M, TM) \dashrightarrow (\mathbb{T}\mathfrak{g}^*, E_{\mathfrak{g}^*})$ is a Hamiltonian $\mathfrak{g}$-space.

Example

A Dirac morphism $(\Phi, \omega)\colon (\mathbb{T}M, TM) \dashrightarrow ((\mathbb{T}G)_\eta, E_G)$ is a quasi-Hamiltonian $\mathfrak{g}$-space.

Example

A Dirac morphism $(\Phi, \omega)\colon (\mathbb{T}M, TM) \dashrightarrow (\mathbb{T}\mathcal{A},\ E_{\mathcal{A}})$ is a Hamiltonian $L\mathfrak{g}$-space.

Reduction of morphisms

Our results on reductions of Dirac structures extend to morphisms.
$\Rightarrow$ recover the correspondence

$$\text{Hamiltonian } LG\text{-spaces} \leftrightarrow \text{q-Hamiltonian } G\text{-spaces}$$

Further directions, open questions

- How to explain the quasi-Poisson structure on G by reduction?
- How to explain the volume forms on q-Hamiltonian spaces by reduction?
- Metric and Kähler aspects?

Many of the foundations of Poisson/Dirac geometry and Lie algebroids in infinite dimensions remain to be developed. (E.g., forthcoming work with Bursztyn and Lima.)