Equivariant indices of Spin^c-Dirac operators for proper moment maps

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Let

- G = compact, connected Lie group;
- M = even-dimensional, connected, Spin^c-manifold with G-action;
- S_M = G-equivariant, $\mathbb{Z}_2$-graded spinor bundle associated to the Spin^c-structure on M.

Let

$$D_M : L^2(M, S_M) \to L^2(M, S_M)$$

be a G-equivariant Spin^c-Dirac operator on M. If M is compact, the G-index

$$\mathrm{Ind}_G(M) := [\ker(D_M^+)] - [\ker(D_M^-)] \in R(G).$$

Paradan-Vergne (2014) give a geometric formula of multiplicities of G-representations in $\mathrm{Ind}_G(M)$. This is a version of the quantization commutes with reduction theorem for Spin^c-manifolds.

The $[Q, R] = 0$ theorem was conjectured and proved for Kähler manifolds by Guillemin-Sternberg (1982). The conjecture for symplectic manifolds was first proved by Meinrenken, Meinrenken-Sjamaar (1996), and then re-proved by Tian-Zhang(1998), Paradan (2001) in different approaches.

Main results:

1. We define a G-equivariant index for possibly non-compact Spin^c-manifolds with proper moment map;
2. show the G-index satisfies the multiplicative property;
3. the G-index decomposes into irreducible representations as in Paradan-Vergne's formula.

For symplectic manifolds, this is the Ma-Zhang's theorem (2008) which generalizes the Vergne conjecture (2006). Later, Paradan (2011) gives a different proof.

Let K_M be the canonical line bundle associated to the Spin^c-structure on M. That is,

$$K_M := \mathrm{Hom}_{\mathrm{Cliff}(TM)}(\overline{S_M}, S_M).$$

A connection ∇^{K_M} on K_M induces a moment map

$$\mu : M \to \mathfrak{g}^* \cong \mathfrak{g}$$

by

$$\sqrt{-1}\langle \mu, \xi \rangle = 2(\mathcal{L}_\xi|_{K_M} - \nabla^{K_M}_{X^\xi}).$$

The moment map μ induces a vector field on M by

$$X^\mu(m) = \frac{d}{dt}\Big|_{t=0} \exp(-t \cdot \mu(m)) \cdot m.$$

We say that the map μ is taming over M if the vanishing set $\{X^\mu = 0\} \subseteq M$ is compact.

Consider the deformed Dirac operator

$$D_f^\mu = D - \sqrt{-1} f \cdot c(X^\mu),$$

where $f : M \to \mathbb{R}^+$ is an admissible function (growth fast enough at infinity). This is the Tian-Zhang's deformed operator generalized by Braverman.

If μ is taming over M, then D_f^μ restricted to

$$D_f^\mu : \left[L^2(M, S_M)\right]^\pi \to \left[L^2(M, S_M)\right]^\pi, \qquad \pi \in \widehat{G}$$

is Fredholm, and has a finite index $\mathrm{Ind}_\pi(D_f^\mu) \in \mathbb{Z}$.

Defines an equivariant index for (M, μ)

$$\mathrm{Ind}_G(M, \mu) := \sum_{\pi \in \widehat{G}} \mathrm{Ind}_\pi(D_f^\mu) \cdot \pi \in \widehat{R}(G).$$

Theorem (Braverman)

1. *The G-index $\mathrm{Ind}_G(M, \mu)$ doesn't depend on the choice of function f.*
2. *The index has the excision property.*
3. *If we have a family of maps $\mu^t : M \times [0, 1] \to \mathfrak{g}$ such that μ^t is taming over $M \times [0, 1]$, then*

$$\mathrm{Ind}_G(M, \mu^0) = \mathrm{Ind}_G(M, \mu^1) \in \widehat{R}(G).$$

The $\mathrm{Ind}_G(M, \mu)$ coincides with the index defined using transversally elliptic operator (Paradan, Paradan-Vergne) or APS-index (Ma-Zhang).

We no longer assume that the moment map μ on M is taming but proper. Later we will show that proper is a weaker condition than taming.

We have a decomposition of the vanishing set

$$\{X^\mu = 0\} = \bigsqcup_{\alpha \in \Gamma} G \cdot \big(M^\alpha \cap \mu^{-1}(\alpha)\big).$$

Here $\Gamma \subset \mathfrak{t}_+$ and for all $R > 0$, there are at most finitely many $\alpha \in \Gamma$ with $\|\alpha\| \leq R$.

Let $\{U_\alpha\}_{\alpha \in \Gamma}$ be a collection of small, disjoint G-invariant open subset of M such that

$$G \cdot \left(M^\alpha \cap \mu^{-1}(\alpha)\right) \subseteq U_\alpha.$$

Theorem (Hochs-S)

For any Spin^c-manifold M with proper moment map, one can define

$$\mathrm{Ind}_G(M, \mu) := \sum_{\alpha \in \Gamma} \mathrm{Ind}_G(U_\alpha, \mu|_{U_\alpha}) \in \widehat{R}(G).$$

Theorem (Hochs-S)

For every irreducible representation $\pi \in \widehat{G}$, there is a constant C_π such that for every G-equivariant Spin^c-manifold M, with taming moment map μ, one has that

$$\left[\mathrm{Ind}_G(M, \mu) : \pi\right] = 0$$

if $\|\mu(m)\| > C_\pi$ for all $m \in M$.

For example, $C_\pi = \|\rho\|$ for trivial G-representation.

The square of the Tian-Zhang's deformed operator

$$D_T^{\mu,2} = D^2 - \sqrt{-1}\,T \sum_{j=1}^{\dim M} c(e_j)c(\nabla_{e_j}^{TM} X^\mu) + 2\sqrt{-1}\,T \nabla_{X^\mu}^{S_M} + T^2 \cdot \|X^\mu\|^2.$$

Notice that

$$S_M = S(TM) \otimes K_M^{\frac{1}{2}}, \qquad X^\mu = \sum_{j=1}^{\dim G} \mu_j \cdot V_j,$$

one has that

$$\nabla_{X^\mu}^{S_M} = \mathcal{L}_\mu - \sqrt{-1}\|\mu\|^2 + \frac{1}{4} \sum_{j=1}^{\dim M} c(e_j)c(\nabla_{e_j}^{TM} X^\mu) - \frac{1}{4} \sum_{j=1}^{\dim G} c((d\mu_j)^*)c(V_j). \tag{1}$$

Each U_α has the geometric structure

$$U_\alpha = G \times_{G_\alpha} V_\alpha.$$

We decompose $G_\alpha = A \cdot H$ with $\mathfrak{g}_\alpha = \mathfrak{a} \oplus \mathfrak{h}$. Accordingly

$$\mu|_{V^\alpha} = \mu_\mathfrak{a} + \mu_\mathfrak{h}.$$

For any $t \in (0, 1]$, define

$$\mu^t|_{V^\alpha} = \mu_\mathfrak{a} + t \cdot \mu_\mathfrak{h}.$$

It extends to a G-equivariant map on U_α.

Lemma

There exists U_α such that

$$\{X^{\mu^t} = 0\} = \{X^{\mu} = 0\} \subseteq U_\alpha$$

for all $t \in (0, 1]$.

This implies that

$$\mathrm{Ind}_G(U_\alpha, \mu) = \mathrm{Ind}_G(U_\alpha, \mu^t), \qquad t \in (0, 1].$$

Proposition

For any $\epsilon > 0$, there is a small open neighborhood U_α of m, and constants $C, \delta > 0$ such that for all $t \in (0, \delta)$

$$D_T^{\mu^t,2} \geq \underbrace{D^2 - T \cdot \mathrm{Tr}|\mathcal{L}_{\mu^t}^{T_m M}| + T^2 \cdot \|X^{\mu^t}\|^2}_{\text{Harmonic oscillator}}$$

$$+ T \cdot \big(\underbrace{2\|\alpha\|^2 + \frac{1}{2}\mathrm{Tr}|\mathcal{L}_\alpha^{T_m M}| - \mathrm{Tr}|\mathcal{L}_\alpha^{\mathfrak{g}/\mathfrak{g}_\alpha}|}_{d(m)} - \epsilon\big) \qquad (2)$$

$$+ 2T\sqrt{-1}\mathcal{L}_{\mu^t} - C.$$

This leads to the vanishing result.

The function $d(m)$ plays an important role in the work of Paradan-Vergne. It is interesting to see it arises in an analytic way.

Theorem

Let M be a complete, Spin^c G-manifold with a proper moment map μ^M, and N is a compact Spin^c G-manifold, then

$$\mathrm{Ind}_G(M \times N, \mu^{M\times N}) = \mathrm{Ind}_G(M, \mu^M) \otimes \mathrm{Ind}_G(N, \mu^N) \in \widehat{R}(G).$$

The multiplicative property is equivalent to

$$\mathrm{Ind}_G(M \times N, \mu^M + \mu^N) = \mathrm{Ind}_G(M \times N, \mu^M) \in \widehat{R}(G).$$

The map

$$\phi^t = \mu^M + t \cdot \mu^N : M \times N \times [0,1] \to \mathfrak{g}$$

might NOT be taming over $M \times N \times [0,1]$. That is, the linear path between two taming maps need not be taming itself, so that certainly not all taming maps are homotopic.

Lemma

Suppose that μ is taming over U and

$$\{X^\mu = 0\} \subset V \subset U$$

be a G-invariant, relatively compact open subset. Then we can choose a moment map $\tilde{\mu} : U \to \mathfrak{g}$ with the following properties:

- $\tilde{\mu}$ *is proper;*
- $\tilde{\mu}|_V = \mu|_V$;
- $\|\tilde{\mu}\| \geq \|\mu\|$;
- *the vector fields X^μ and $X^{\tilde{\mu}}$ have the same set of zeroes.*

The moment map μ is given by formula

$$\sqrt{-1}\langle \mu, \xi \rangle = 2(\mathcal{L}_\xi|_{K_M} - \nabla^{K_M}_{X^\xi}).$$

Define a new connection

$$\widetilde{\nabla}^{K_M} = \nabla^{K_M} - \sqrt{-1} \cdot f \cdot (X^\mu)^*.$$

The new moment map

$$\sqrt{-1}\langle \tilde{\mu}, \xi \rangle = 2(\mathcal{L}_\xi|_{K_M} - \widetilde{\nabla}^{K_M}_{X^\xi}).$$

We fix an irreducible G-representation π.

Let

$$U_{M\times N} \subset M \times N, \quad U_M \subset M$$

be G-invariant, relative compact, open subsets so that

$$\left[\mathrm{Ind}_G(U_{M\times N}, \mu^{M\times N}) : \pi\right] = \left[\mathrm{Ind}_G(M \times N, \mu^{M\times N}) : \pi\right],$$

and

$$\left[\mathrm{Ind}_G(U_M \times N, \mu^{M}) : \pi\right] = \left[\mathrm{Ind}_G(M \times N, \mu^{M}) : \pi\right].$$

Strategy of the proof:

$$\begin{aligned}\left[\mathrm{Ind}_G(M \times N, \mu^M) : \pi\right] &= \left[\mathrm{Ind}_G(U_{M\times N}, \mu^M) : \pi\right] \\ &= \left[\mathrm{Ind}_G(U_{M\times N}, \tilde{\mu}^M + \mu^N) : \pi\right] \\ &= \left[\mathrm{Ind}_G(U_M \times N, \mu^M + \mu^N) : \pi\right] \\ &= \left[\mathrm{Ind}_G(M \times N, \mu^M + \mu^N) : \pi\right]\end{aligned} \tag{3}$$

For the equation

$$\mathrm{Ind}_G(U_{M\times N}, \tilde{\mu}^M + \mu^N) = \mathrm{Ind}_G(U_{M\times N}, \tilde{\mu}^M),$$

we consider

$$\phi^t : U_{M\times N} \times [0,1] \to \mathfrak{g} : (m,n) \mapsto \tilde{\mu}^M(m) + t \cdot \mu^N(n).$$

Lemma

The map ϕ^t is taming over $U_{M\times N} \times [0,1]$. That is, $\{X^{\phi^t} = 0\}$ is compact.

This also implies that the index

$$\mathrm{Ind}_G(U_{M\times N}, \tilde{\mu}^M + \mu^N) \in \widehat{R}(G)$$

doesn't depend on how you make μ^M proper.

For the second equation, we choose a relative compact open subset

$$V \subset U_{M \times N} \cap (U_M \times N).$$

We choose the function f so that

$$\tilde{\mu}^M | V = \mu^M | V.$$

By the vanishing result,

$$\left[\mathrm{Ind}_G(U_{M \times N}, \tilde{\mu}^M + \mu^N) : \pi\right] = \left[\mathrm{Ind}_G(U_M \times N, \mu^M + \mu^N) : \pi\right].$$

Let $\mathcal{O} = G \cdot \rho \cong G/T$ be a coadjoint orbit of G. In particular, $\mathrm{Ind}_G(\mathcal{O})$ gives the trivial representation. It follows that

$$\left[\mathrm{Ind}_G(M, \mu^M)\right]^G = \left[\mathrm{Ind}_G(M \times (-\mathcal{O}), \mu^{M\times(-\mathcal{O})})\right]^G.$$

The index localizes to the vanishing set

$$Z_{M\times(-\mathcal{O})} = \{X^{\mu^{M\times(-\mathcal{O})}} = 0\} = \bigsqcup_{\alpha\in\Gamma} Z_\alpha.$$

Paradan-Vergne show that the function d on $Z_{M\times(-\mathcal{O})}$ is locally constant and non-negative. This reduces the problem to an arbitrary small open neighborhood of $Z^{=0}_{M\times(-\mathcal{O})}$.

Theorem (Hochs-S)

Let G be a compact, connected Lie group, and let M be a even-dimensional, possibly non-compact manifold, with an action by G, and a G-equivariant Spin^c-structure. Let μ be a Spin^c-moment map, and suppose it is proper. If the minimal stabilzer is abelian, then

$$\mathrm{Ind}_G(M, \mu) = \sum_{\pi_\lambda \in \widehat{G}} \mathrm{Ind}(M_{\lambda+\rho}) \cdot \pi_\lambda \in \widehat{R}(G).$$

In general, the multiplicity of each π_λ is a finite sum of indices on reduced spaces.

The End

Thank You