Norm-square localization for Hamiltonian LG-spaces

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- M Hamiltonian G-space, proper moment map $\mu : M \to \mathfrak{g}^*$
- Norm $|\cdot|$ on $\mathfrak{g}^*$

Definition

$|\mu|^2 : M \to \mathbb{R}$ is the **norm-square of the moment map**.

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Some past work:

- Kirwan (1984): Morse theory with $|\mu|^2$, Kirwan surjectivity
- Witten (1992): certain integrals localize to Crit($|\mu|^2$)
- Paradan (1999, 2000): detailed norm-square localization formula
- Woodward (2005), Harada-Karshon (2012): other approaches

Hamiltonian LG-spaces

- G compact Lie group, Lie algebra $\mathfrak{g}$
- Ad-invariant inner product $\langle -, - \rangle$
- $LG = \mathrm{Map}(S^1, G)$ loop group
- $L\mathfrak{g}^* = \Omega^1(S^1, \mathfrak{g})$, LG acts by gauge transformations:

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Definition

A *Hamiltonian LG-space* $(\mathcal{M}, \omega_{\mathcal{M}}, \Psi)$ consists of a symplectic Banach manifold, equipped with an LG-action, and with a proper moment map $\Psi : \mathcal{M} \to L\mathfrak{g}^*$, equivariant for the gauge action of LG on $L\mathfrak{g}^*$.

Similar to finite dimensions: convexity theorem, cross-sections, etc.

Norm-square of Ψ

L^2 norm on $L\mathfrak{g}^*$ by integration: $||\gamma||^2 = \frac{1}{2\pi}\int_0^{2\pi}|\gamma(\theta)|^2 d\theta$.

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Theorem (Kirwan, Bott-Tolman-Weitsman)

The critical set of $||\Psi||^2$ is

$$\mathrm{Crit}(||\Psi||^2) = \bigcup_{\beta \in \mathcal{B}} G \cdot (\mathcal{M}^\beta \cap \Psi^{-1}(\beta)),$$

where $\mathcal{B} \subset \mathfrak{t}_+^*$ is a discrete subset.

Quasi-Hamiltonian G-spaces

- $\theta^L = g^{-1}dg, \theta^R = dgg^{-1}$ the left, resp. right Maurer-Cartan forms on G
- $\eta = \frac{1}{12}\langle[\theta^L, \theta^L], \theta^L\rangle$ Cartan 3-form

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Definition (Alekseev, Malkin, Meinrenken)

A *quasi-Hamiltonian G-space* (M, ω, Φ) is a G-manifold M, equipped with a G-invariant 2-form ω together with an equivariant map $\Phi : M \to G$ satisfying

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Reduction: at conjugacy class $\mathcal{C} \subset G$: $\Phi^{-1}(\mathcal{C})/G$, is **symplectic**.
Examples: conjugacy classes, moduli spaces of flat connections on Riemann surfaces, even-dimensional spheres.

1-1 Correspondence

Theorem (Alekseev, Malkin, Meinrenken)

There is a 1-1 correspondence between compact quasi-Hamiltonian G-spaces (M, ω, Φ), and proper Hamiltonian LG-spaces $(\mathcal{M}, \omega_{\mathcal{M}}, \Psi)$.

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$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\Psi} & L\mathfrak{g}^* \\ \downarrow /L_0G & & \downarrow /L_0G \\ M & \xrightarrow{\Phi} & G \end{array}$$

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- $\omega_{\mathcal{M}}$ *not* basic—must be modified to $\omega_{\mathcal{M}} - \Psi^*\varpi$, which then descends to M.
- Reduced spaces agree, e.g. $\Psi^{-1}(0)/G \simeq \Phi^{-1}(e)/G$.

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Recall:	$\mathrm{Spin}(V) \subset \mathrm{Cliff}(V)$,	$\mathrm{Cliff}(V) \simeq \bigwedge V$.

Compose to get map $\psi : G \to \wedge\mathfrak{g}$, i.e. a left-invariant differential form on G.

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Theorem (Alekseev, Meinrenken, Woodward)

Suppose G simply connected. The top degree form

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Example

G simple, simply connected. Fundamental alcove $\mathfrak{A} \subset \mathfrak{t}_+$.
Conjugacy class $\mathcal{C}_\mu$ containing $\exp(\mu)$, $\mu \in \mathfrak{A}$.

$$\mathrm{Vol}(\mathcal{C}_\mu) = \mathrm{vol}(G/G_{\exp(\mu)}) \prod_{\alpha>0, \langle\alpha,\mu\rangle \notin \mathbb{Z}} 2\sin(\pi\langle\alpha,\mu\rangle).$$

Duistermaat-Heckman distributions

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$$\mathrm{vol}(\Phi^{-1}(e)/G) = \frac{d}{\mathrm{vol}(G)} \frac{\Phi_*|\Gamma|}{d\mathrm{vol}_G}\Big|_e.$$

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More general DH distributions twisted by $\alpha \in H_G(M)$ encode cohomology pairings on quotients.

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Next define related distribution on $T \subset G$.

The map R_G

- $W = N(T)/T$ Weyl group
- ρ half-sum pos. roots, n_+ number of pos. roots
- χ_λ irreducible rep. corresponding to dominant weight λ

Definition

There is an isomorphism

$$R_G : \mathcal{D}'(G)^G \xrightarrow{\sim} \mathcal{D}'(T)^{W-\text{anti}}.$$

determined by the equation

$$i^{-n_+}\langle \mathfrak{n}, \overline{\chi_\lambda}\rangle = \mathrm{vol}_{G/T}\langle R_G(\mathfrak{n}), \overline{t^{\lambda+\rho}}\rangle.$$

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Example

Let $f \in C^\infty(G)^G$, and $J(t) = \sum_{w\in W}(-1)^{|w|}t^\rho$.

$$R_G(f\ d\mathrm{vol}_G) = J \cdot (f|_T)\ d\mathrm{vol}_T.$$

Some conjugacy classes

Example

Let $\mu \in \mathrm{int}(\mathfrak{A})$ and

$$\iota : \mathcal{C}_\mu \hookrightarrow G.$$

$\iota_*|\Gamma| \in \mathcal{D}'(G)^G$ is a delta distribution on $\mathcal{C}_\mu \subset G$ with total weight $\mathrm{Vol}(\mathcal{C}_\mu)$. Then:

$$R_G(\iota_*|\Gamma|) = \frac{1}{|W|} \sum_{w \in W} (-1)^{|w|} \delta_{w \exp \mu}.$$

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Example

$e \in G$ identity element

$$R_G(\delta_e^G) = \left(\prod_{\alpha < 0} \partial_\alpha \right) \delta_e^T.$$

A DH distribution for $\Phi^{-1}(T)$

Definition

Recall $\Phi_*|\Gamma| \in \mathcal{D}'(G)^G$ is the DH distribution of M. We define

$$\mathfrak{m} = R_G(\Phi_*|\Gamma|) \in \mathcal{D}'(T)^{W-\text{anti}}.$$

$\mathfrak{m}$ is the distribution we will discuss for the rest of the talk.

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Basic properties:

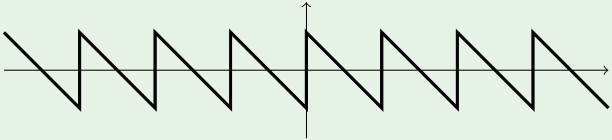
- Φ transverse to $T \Rightarrow \mathfrak{m}$ is DH distribution of $\Phi^{-1}(T) \subset M$.
- Gives $\mathrm{vol}(\Phi^{-1}(\mathcal{C}_\mu)/G)$ directly for $\mu \in \mathrm{int}(\mathfrak{A})$.
- If Φ has regular values, it is piecewise-polynomial.

Examples of $\mathfrak{m} \in \mathcal{D}'(T)^{W-\text{anti}}$

Example

$$DSU(2) = SU(2) \times SU(2) \circlearrowleft SU(2), \qquad \Phi(a,b) = aba^{-1}b^{-1}$$

$$|\Gamma| = \text{Haar measure} \qquad \mathfrak{m} = R_G(\Phi_*|\Gamma|) \text{ is:}$$



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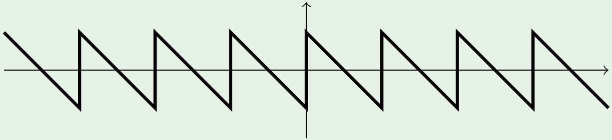
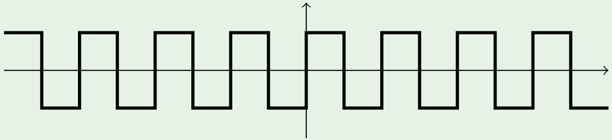


Example

$$S^4 \circlearrowleft SU(2) \qquad \mathfrak{m} = R_G(\Phi_*|\Gamma|) \text{ is:}$$



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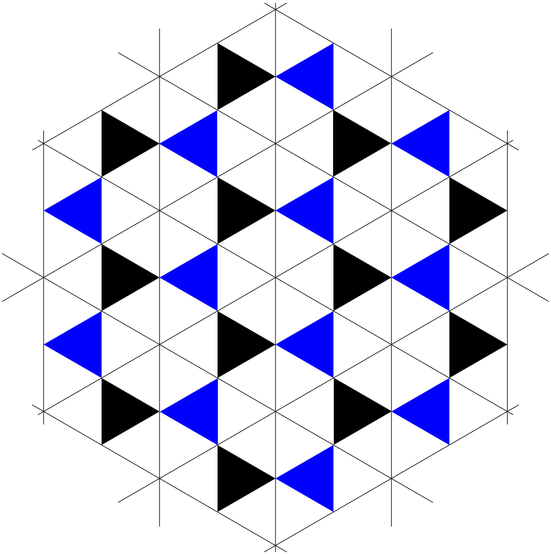


Example

Multiplicity-free, quasi-Hamiltonian $SU(3)$-space (Chris Woodward).

Main result

Recall: $\mathrm{Crit}(||\Psi||^2) = \bigcup_{\beta \in \mathcal{B}} G \cdot (\mathcal{M}^\beta \cap \Psi^{-1}(\beta))$

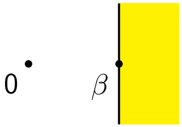
Theorem

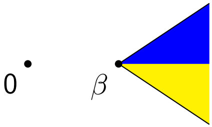
There is a norm-square localization formula for $\mathfrak{m}$*:*

$$\mathfrak{m} = \sum_{\beta \in W \cdot \mathcal{B}} \mathfrak{m}_\beta .$$

- $\mathfrak{m}_\beta$ *piecewise-polynomial on cones with apex at* β*.*
- $\beta \neq 0 \Rightarrow$ *support* $\mathfrak{m}_\beta$ *contained in half-space* $\beta \geq ||\beta||^2$*.*
- *Expression for* $\mathfrak{m}_\beta$ *as integral over submanifold near* $\mathcal{M}^\beta \cap \Psi^{-1}(\beta)$*, involving local geometric data.*

$\mathfrak{m}_\beta$





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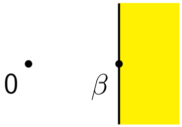

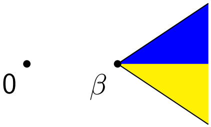

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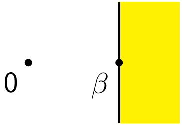

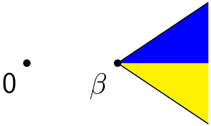

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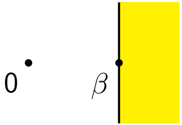
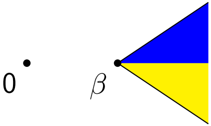
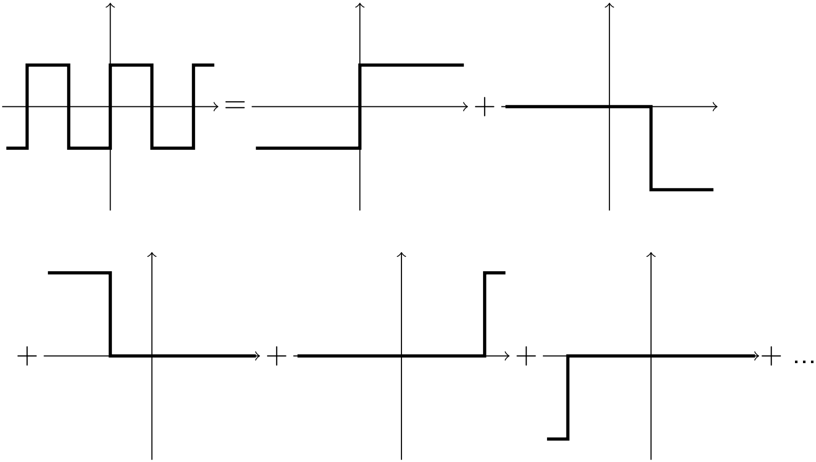


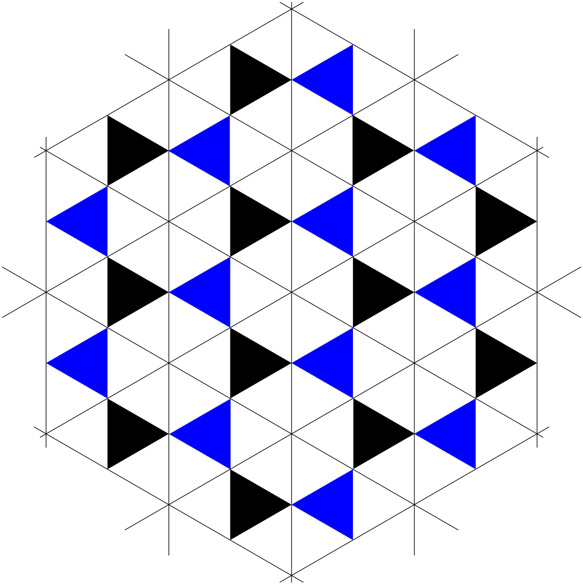


- $\phi : Y \to U$ cross-section, $\beta \in U \subset L\mathfrak{g}^*$, project $\phi_\mathfrak{t} := \mathrm{pr}_{\mathfrak{t}^*} \circ \phi$.
- Minimal coupling $\to$ $\mathrm{Tot}(\nu(Y^\beta, Y))$ becomes Hamiltonian T-space (**polarized** weights).
- For $\mathfrak{m}_\beta$: take **germ** of (twisted) DH measure for $\mathrm{Tot}(\nu(Y^\beta, Y))$ near β and **extend**.
- Explicit formula similar to Paradan (2000). Convolutions of polynomial distributions on walls, and Heaviside distributions H_α, $\langle \alpha, \beta \rangle > 0$.

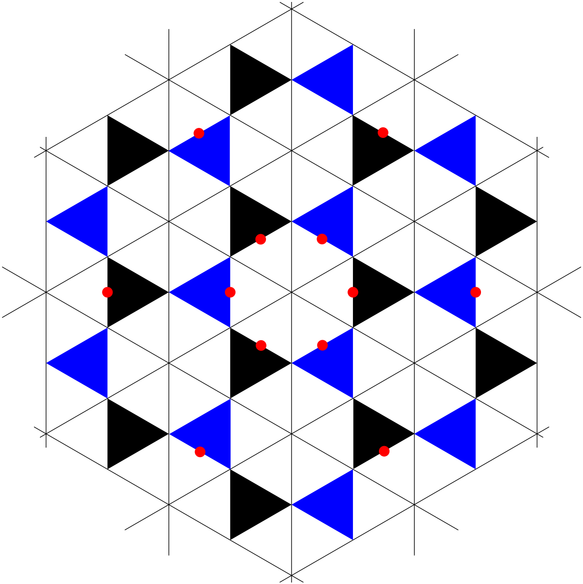
Norm-square localization for S^4



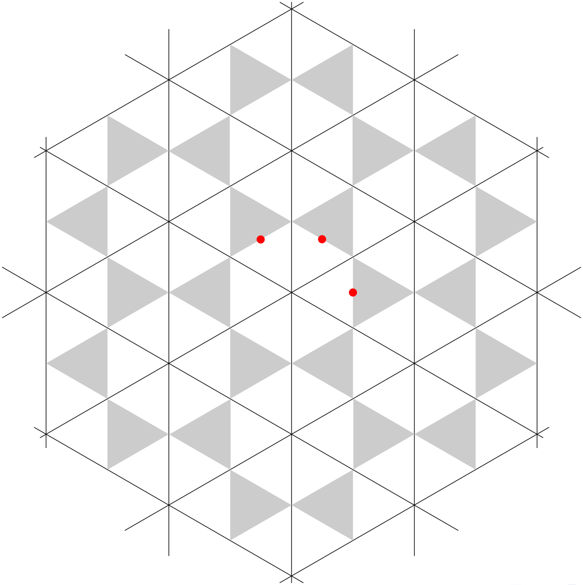
Chris Woodward's example



The set $W \cdot \mathcal{B}$



First three contributions



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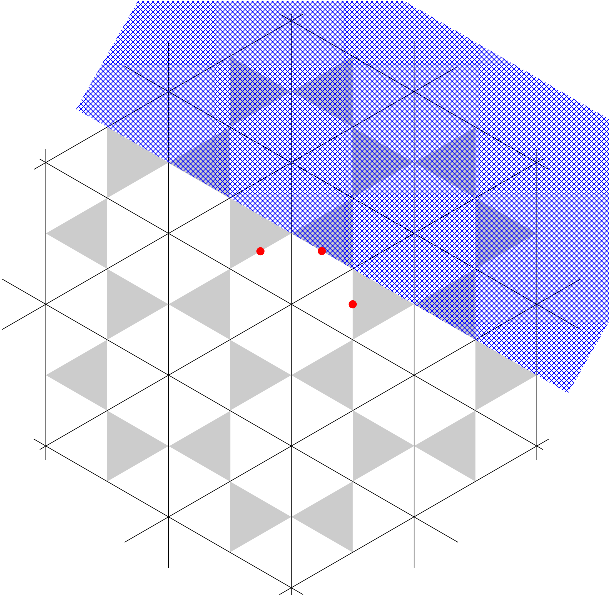

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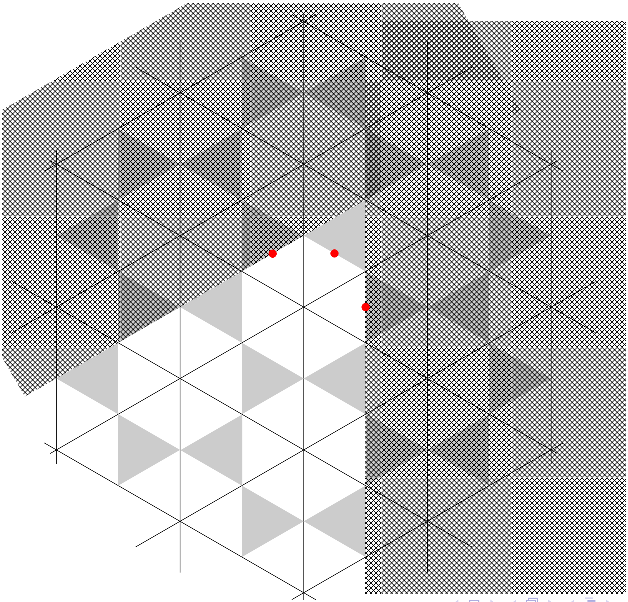

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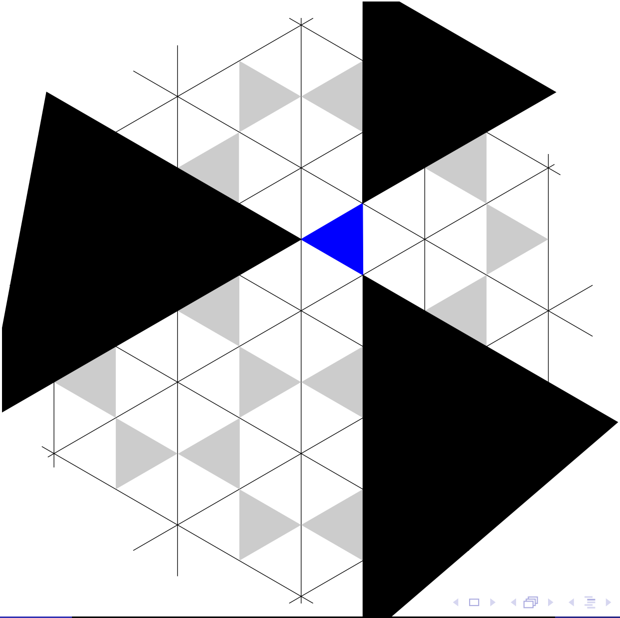

Outline of method

Strategy modelled on that of Szenes-Vergne ("[Q,R]=0 and Kostant multiplicity functions", 2010).

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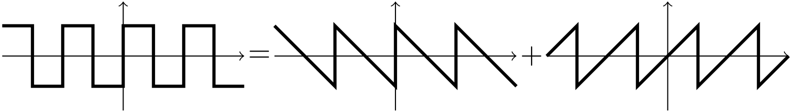
For explicit expressions (similar to Paradan (2000)) involving an integral near the critical set, some additional argument involved (more technical, not for today).

Example: S^4



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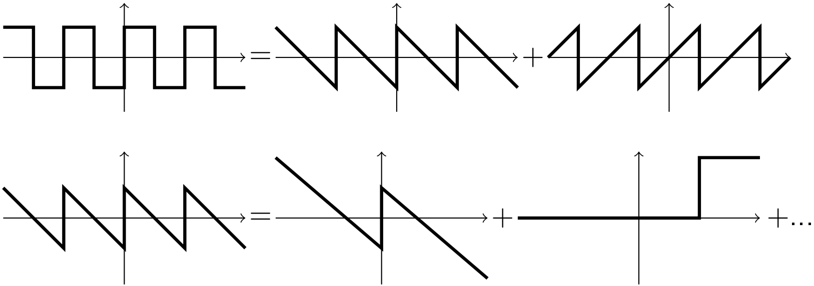

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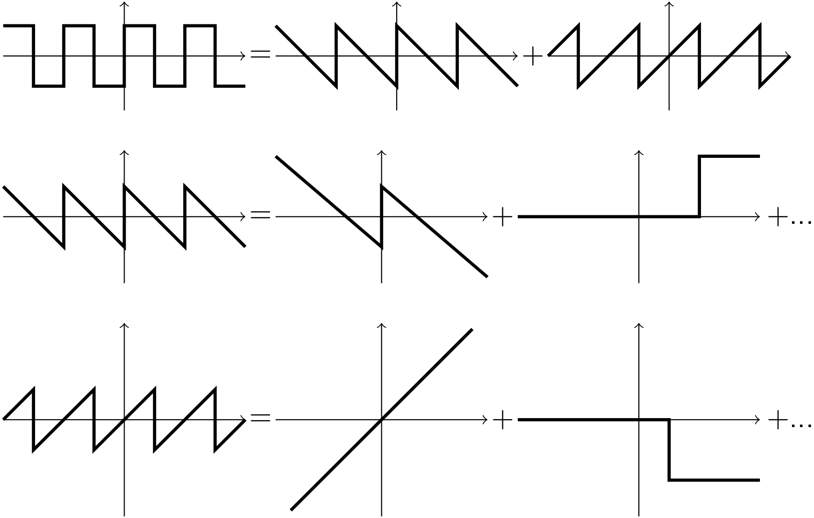
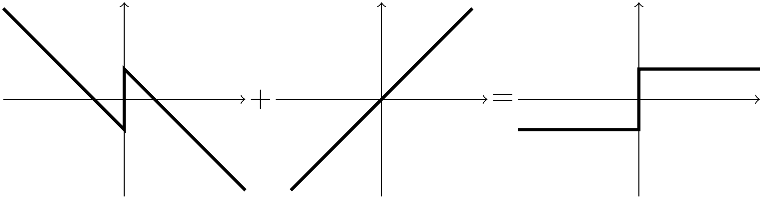
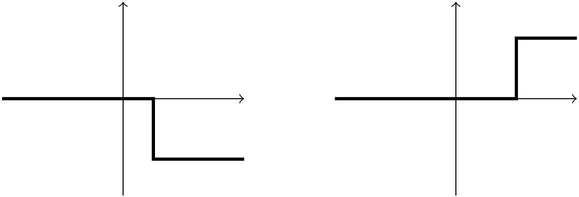


Example: S^4

Add central contributions:



Next two contributions (along positive $\mathbb{R}$ axis):



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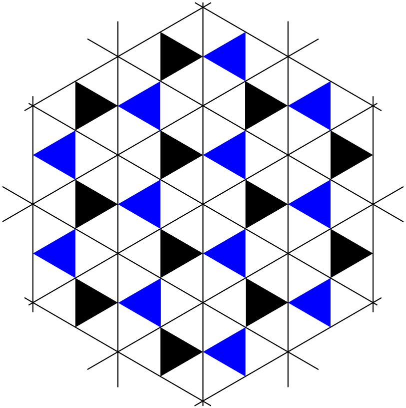

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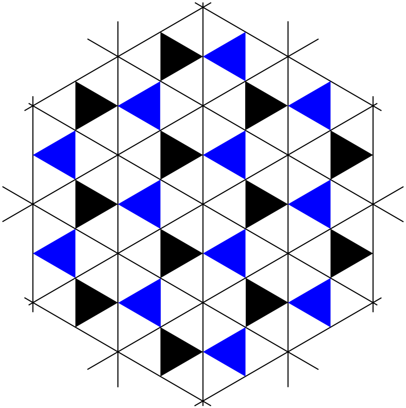
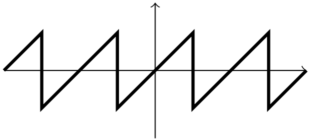


3 Fixed-point contributions: pullbacks of



Central contribution from each is a non-trivial linear function.

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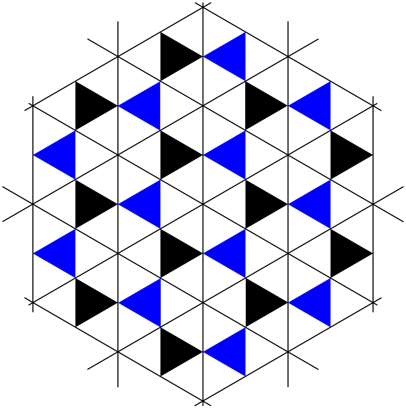

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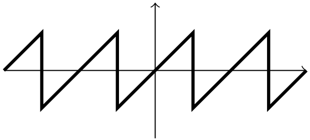

Central contribution from each is a non-trivial linear function. But the **sum** is **zero**.

Abelian localization formula for $\mathfrak{m}$

$\rightarrow$ Identify $\mathfrak{t} \simeq \mathfrak{t}^*$.

Theorem (Alekseev, Meinrenken, Woodward)

Let $\xi \in \Lambda^*$ (weight lattice). We have the following abelian localization formula for the Fourier coefficients of $\mathfrak{m}$:

$$\langle \mathfrak{m}, t^\xi \rangle = \prod_{\alpha > 0} 2\pi i \langle \alpha, \xi \rangle \sum_{F \subset M^\xi} \int_F \frac{e^{2\pi i \omega} \Phi^\xi}{\mathrm{Eul}(\nu_F, \xi)}.$$

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Fourier inversion:

$$\mathfrak{m}(\mu) = \prod_{\alpha<0} \partial_\alpha \sum_{\xi \in \Lambda^*} \sum_{F \subset M^\xi} \int_F \frac{e^{2\pi i \omega} \Phi^\xi}{\mathrm{Eul}(\nu_F, \xi)} e^{-2\pi i \langle \mu, \xi \rangle}.$$

Want to interchange the two summations.

Abelian localization formula for $\mathfrak{m}$

Take F, closure of a T-orbit type.

- $\mathfrak{t}_F \subset \mathfrak{t}$ infinitesimal stabilizer of F
- $\alpha_i, i = 1, ..., N$ list of weights on the normal bundle ν_F
- $(\Lambda^* \cap \mathfrak{t}_F) \setminus \cup\{\alpha_i = 0\}$ subset of Λ^* where $\int_F$ appears

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$\Rightarrow$ $\mathfrak{m}_F$ is sum of (shifted) **multiple Bernoulli series**.

Multiple Bernoulli series

- V vector space, $\Gamma \subset V$ lattice, dual $\Gamma^* \subset V^*$
- $\underline{\alpha}$ a list of elements of Γ^*

Definition (Szenes (1998), Brion-Vergne (1999),...)

The *multiple Bernoulli series* associated to the data $V, \Gamma, \underline{\alpha}$ is:

$$B_{\underline{\alpha},\Gamma}(\mu) = \sum_{\xi \in \Gamma}{}' \frac{e^{2\pi i \langle \mu, \xi \rangle}}{\prod_k 2\pi i \langle \alpha_k, \xi \rangle}.$$

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Examples

- $B_{\emptyset,\mathbb{Z}}(x) = \sum_{n \in \mathbb{Z}} e^{2\pi i n x} = \sum_{n \in \mathbb{Z}} \delta(x - n)$
- $B_{1,\mathbb{Z}}(x) = \sum_{n \neq 0} \frac{e^{2\pi i n x}}{2\pi i n} = \frac{1}{2} - x + \lfloor x \rfloor$

Boysal-Vergne decomposition formula

$\rightarrow$ Choose generic $\gamma \in V^*$, and inner product.

Theorem (Boysal-Vergne)

There is a decomposition:

$$B_{\underline{\alpha},\Gamma} = \sum_{\Delta \in \mathcal{A}} B_{\underline{\alpha},\Gamma,\Delta},$$

where

- *$\mathcal{A}$ an infinite collection of affine subspaces $\Delta \subset V^*$,*
- *$B_{\underline{\alpha},\Gamma,\Delta}$ is a convolution of a polynomial distribution on Δ with Heaviside distributions in transverse directions,*
- *for $\Delta \neq V^*$, $\gamma \notin support(B_{\underline{\alpha},\Gamma,\Delta})$.*

Decomposition of $\mathfrak{m}_F$

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- Apply Boysal-Vergne type decomposition.
- Group terms according to affine subspaces Δ.
- Subalgebra $\Delta^\perp =: \mathfrak{t}_\Delta \subset \mathfrak{t}$.
- Further grouping according to lattice of T orbit-types.

Then:

$$\mathfrak{m} = \sum_\Delta \sum_{C\subset M^{\mathfrak{t}_\Delta}} \mathfrak{m}_{\Delta,C}.$$

Reinterpretation of summands

- $C \subset M^{t_\Delta}$ is a *quasi-Hamiltonian* space.
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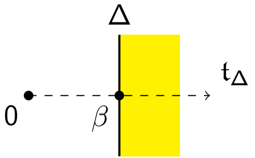

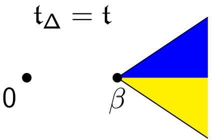

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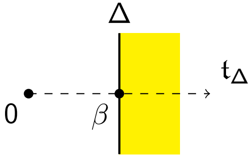
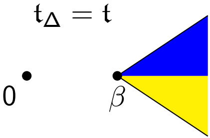
$\Rightarrow$ Term $\mathfrak{m}_{\Delta,C}$ corresponding to $C \subset M^{\mathfrak{t}_\Delta}$ is the extension of the *germ* (at β) of a DH distribution for $\mathrm{Tot}(\nu(C, M))$! (abelian localization on $\mathrm{Tot}(\nu(C, M))$)

Reinterpretation of summands





- $\Rightarrow$ contribution vanishes unless $\Phi^{-1}(\exp(\beta)) \cap M^{\mathfrak{t}_\Delta} \neq \emptyset$. Since

$$\Phi^{-1}(\exp(\beta)) \cap M^{\mathfrak{t}_\Delta} \simeq \Psi^{-1}(\beta) \cap \mathcal{M}^{\mathfrak{t}_\Delta},$$

  $\Rightarrow$ non-zero contributions indexed by components of critical set of $||\Psi||^2$.

- Further argument leads to explicit formulas involving integrals in cross-sections near critical set.