$[Q, R] = 0$ for spin^c Dirac operators

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Adelaide, July 28, 2015

Spin^c Dirac operators

Let M be a compact, oriented and even dimensional manifold. Let $Cl(TM) \to M$ be the Clifford bundle associated to a Riemannian metric.

Clifford module

A complex vector bundle $\mathcal{E} \to M$ is a $Cl(TM)$-module if there is a bundle map $\mathbf{c}_{\mathcal{E}} : TM \to End(\mathcal{E})$ such that

$$\mathbf{c}_{\mathcal{E}}(v)^2 = -\|v\|^2 \mathrm{Id}_{\mathcal{E}} \quad \text{for all} \quad v \in TM.$$

Spinor bundle

- A spinor bundle $\mathcal{S} \to M$ is an *irreducible* $Cl(TM)$-module.
- The orientation induces a grading $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ such that $\mathbf{c}_{\mathcal{S}}(v)$ are odd endomorphisms.

We can associate to a spinor bundle $\mathcal{S} \to M$ a Dirac operator

$$D_{\mathcal{S}} : \Gamma(M, \mathcal{S}^+) \to \Gamma(M, \mathcal{S}^-).$$

Since $D_{\mathcal{S}}$ is elliptic we may consider its index

$$\mathcal{Q}(M, \mathcal{S}) := \mathrm{Index}(D_{\mathcal{S}}) \ \in \ \mathbb{Z}.$$

Atiyah-Singer formula

We have

$$\mathcal{Q}(M, \mathcal{S}) = \int_M e^{i\Omega_{\mathcal{S}}} \widehat{\mathrm{A}}(M),$$

where $\Omega_{\mathcal{S}}$ is **half** the curvature of the line bundle

$$\det(\mathcal{S}) := \hom_{Cl(TM)}(\overline{\mathcal{S}}, \mathcal{S}).$$

Spin^c Dirac operators: the equivariant case

- Let K be a compact connected Lie group acting on $\mathcal{S} \to M$.

- The Dirac operator $D_{\mathcal{S}}$ is K-equivariant, and its equivariant index $\mathcal{Q}_K(M, \mathcal{S})$ can be computed by the delocalized formulae of Berline-Vergne: for $X \in \mathfrak{k}$ small enough

$$\mathcal{Q}_K(M, \mathcal{S})(e^X) = \int_M e^{i\Omega_{\mathcal{S}}(X)} \widehat{\mathrm{A}}(M, X),$$

where

$$\Omega_{\mathcal{S}}(X) := \Omega_{\mathcal{S}} + \langle \Phi_{\mathcal{S}}, X \rangle$$

is half the **equivariant curvature** of the line bundle $\det(\mathcal{S})$.

Atiyah-Hirzebruch (70's)

If the line bundle $\det(\mathcal{S})$ is trivial, then $\mathcal{Q}_K(M, \mathcal{S}) = 0$ unless the action $K \circlearrowright M$ is trivial.

Parametrization of $\widehat{K}$

Admissible orbits

• A coadjoint orbit $\mathcal{P} \subset \mathfrak{k}^*$ is admissible if there exists an equivariant spinor bundle $\mathcal{S}_\mathcal{P}$ such that $\Phi_{\mathcal{S}_\mathcal{P}}$ is the inclusion.
• We denote $\mathcal{Q}_K^{spin}(\mathcal{P}) := \mathcal{Q}_K(\mathcal{P}, \mathcal{S}_\mathcal{P})$.
• Let $\mathcal{A}$ be the set of admissible orbits, and let $\mathcal{A}_{reg} \subset \mathcal{A}$ be the subset formed by the **regular** orbits.

Facts

• The element $\mathcal{Q}_K^{spin}(\mathcal{P})$ is either 0 or an irred. rep. of K.
• The map $\mathcal{P} \longrightarrow \mathcal{Q}_K^{spin}(\mathcal{P})$ is not injective.

Parametrization

The map $\mathcal{O} \in \mathcal{A}_{reg} \longrightarrow \pi_\mathcal{O} := \mathcal{Q}_K^{spin}(\mathcal{O}) \in \widehat{K}$ is bijective.

Vanishing results

• Let $(\mathfrak{k}_M)$ the generic infinitesimal stabilizer for the K-action on M.

• Let $\mathcal{H}$ be the set of infinitesimal stabilizers $(\mathfrak{k}_\xi), \xi \in \mathfrak{k}$, and let $\mathcal{H}'$ be the set formed by their semi-simple part $([\mathfrak{k}_\xi, \mathfrak{k}_\xi]), \xi \in \mathfrak{k}$.

Theorem 1, P-Vergne

If $([\mathfrak{k}_M, \mathfrak{k}_M]) \notin \mathcal{H}'$, then

$$\mathcal{Q}_K(M, \mathcal{S}) = 0$$

for any spinor bundle $\mathcal{S}$.

Remark

The result above does not hold for more general Dirac operators.

We suppose that $\exists\, \mathfrak{h} \in \mathcal{H}$ such that $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$.

Let $\mathcal{S} \to M$ be an equivariant spinor bundle: the choice of a connection on $\det(\mathcal{S})$ determines an equivariant map $\Phi_{\mathcal{S}} : M \to \mathfrak{k}^*$. Note that

$$\Phi_{\mathcal{S}}(M) \subset \{\xi \in \mathfrak{k}^* \,|\, (\mathfrak{h}) \subset (\mathfrak{k}_\xi)\}.$$

Theorem 2, P-Vergne

If $\mathcal{Q}_K(M, \mathcal{S}) \neq 0$ then

$$\Phi_{\mathcal{S}}^{-1}\left(\{\xi \in \mathfrak{k}^* \,|\, (\mathfrak{h}) = (\mathfrak{k}_\xi)\}\right)$$

is **open** and **dense** in M.

Geometric consequence

The manifold M has a dense open part of the form $K \times_H Y$ where Y is a $H/[H, H]$-submanifold of M.

Multiplicies

• Thanks to the parametrization $\mathcal{O} \in \mathcal{A}_{reg} \mapsto \pi_{\mathcal{O}} \in \widehat{K}$, we define $m_{\mathcal{O}}$ as the multiplicity of $\pi_{\mathcal{O}}$ in $\mathcal{Q}_K(M, \mathcal{S})$.

• Let $\mathcal{A}((\mathfrak{h}))$ be the subset of admissible orbits of type $(\mathfrak{h})$. For any $\mathcal{P} \in \mathcal{A}((\mathfrak{h}))$, we define the reduced space

$$M_{\mathcal{P}} := \Phi_{\mathcal{S}}(\mathcal{P})/K.$$

Theorem 3, P-Vergne

• We have

$$m_{\mathcal{O}} = \sum_{\mathcal{P}} \mathcal{Q}^{spin}(M_{\mathcal{P}})$$

where the sum runs over $\mathcal{P} \in \mathcal{A}((\mathfrak{h}))$ such that $\mathcal{Q}_K^{spin}(\mathcal{P}) = \pi_{\mathcal{O}}$.

• The spin^c indices $\mathcal{Q}^{spin}(M_{\mathcal{P}})$, which are defined by shift desingularization, do not depend on the choice of connection.

Previous works in the spin^c setting

Torus actions

- Y. Karshon and S. Tolman (1993), M. Grossberg and Y. Karshon (1994,1998) : **toric manifolds.**
- A. Cannas da Silva, Y. Karshon and S. Tolman (2000) : **circle actions.**

Non-abelian group actions and $\Omega_{\mathcal{S}}$ is symplectic

- L. Jeffrey and F. Kirwan (1997) : **asymptotic result**
- PEP (2012)

Example : the Hirzebruch surface

Let M be the quotient of $U := \mathbb{C}^2 - \{(0,0)\} \times \mathbb{C}^2 - \{(0,0)\}$ by the free action of $\mathbb{C}^* \times \mathbb{C}^*$ acting by

$$(u,v)\cdot(z_1,z_2,z_3,z_4) = (uz_1, uz_2, uvz_3, vz_4).$$

Consider the **non-ample** line bundle $\mathbb{L}$ obtained as quotient of the trivial line bundle $U \times \mathbb{C} \to U$ by the action

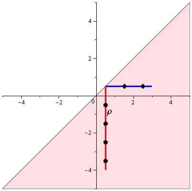
$$(u,v)\cdot(z_1,z_2,z_3,z_4,z) = (uz_1, uz_2, uvz_3, vz_4, u^3v^6z).$$

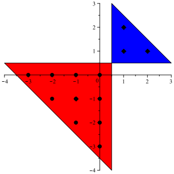
We have a natural holomorphic action of $U(2)$ on $\mathbb{L} \to M$: the Euler characteristic $H^0(M, O(\mathbb{L})) - H^1(M, O(\mathbb{L})) + H^2(M, O(\mathbb{L}))$ is a $U(2)$-representation equal to

$$\mathcal{Q}_{U(2)}(M, \mathcal{S})$$

where $\mathcal{S} = \bigwedge_{\mathbb{C}} \mathrm{T}M \otimes \mathbb{L}$.

In this example we can compute everything and check the validity of our $[Q, R] = 0$ theorem.





Main steps of the proof

The proof of Theorems 1,2,3 can be divided in the following steps:

- Witten deformation
- Fixed point formula for localized indices
- Function $d_{\mathcal{S}}$
- Shifting trick
- Magical inequality

Witten deformation

• Let $\sigma(M,\mathcal{S})(m,v) = \mathbf{c}_m(v) : \mathcal{S}_m^+ \to \mathcal{S}_m^-$ be the the principal symbol of the Dirac operator $D_{\mathcal{S}}$.
• The Kirwan vector field is $\kappa_{\mathcal{S}}(m) = \Phi_{\mathcal{S}}(m) \cdot m$.
• Let $Z_{\mathcal{S}} = \{\kappa_{\mathcal{S}} = 0\}$.
• The symbol $\sigma(M,\mathcal{S})$ pushed by the vector field $\kappa_{\mathcal{S}}$ is

$$\sigma(M,\mathcal{S},\Phi_{\mathcal{S}})(m,v) = \mathbf{c}_m(v + \kappa_{\mathcal{S}}(m)).$$

We have $\mathcal{Q}_K(M,\mathcal{S}) = \mathrm{Index}_K(\sigma(M,\mathcal{S})) = \mathrm{Index}_K(\sigma(M,\mathcal{S},\Phi_{\mathcal{S}}))$.

Basic fact

If $U \subset M$ is an open invariant subset such that $Z := U \cap Z_{\mathcal{S}}$ is compact, then $\sigma(M,\mathcal{S},\Phi_{\mathcal{S}})|_{\mathrm{T}^*U}$ is transversally elliptic. We denote

$$\mathcal{Q}_K(M,\mathcal{S},Z)$$

its equivariant index.

Localization à la Witten

If we have a disjoint decomposition in compact subsets

$$Z_{\mathcal{S}} = \coprod_{i \in I} Z_i$$

the excision property gives

$$\mathcal{Q}_K(M, \mathcal{S}) = \sum_{i \in I} \mathcal{Q}_K(M, \mathcal{S}, Z_i).$$

Question

Determine when $[\mathcal{Q}_K(M, \mathcal{S}, Z_i)]^K \neq 0$.

We can use the finite decomposition $Z_{\mathcal{S}} = \coprod_{\beta} Z_{\beta}$ where

$$Z_{\beta} = K(M^{\beta} \cap \Phi_{\mathcal{S}}^{-1}(\beta)).$$

Fixed point formula for $\mathcal{Q}_K(M, \mathcal{S}, Z_\beta)$ when $\beta \neq 0$

Let $\mathcal{N}$ be the normal bundle of M^β in M: the linear action of β on the fibers induces a **complex structure.**

Fixed point formula à la Atiyah-Segal-Singer

The spinor bundle $\mathcal{S}$ on M induces a spinor bundle $\mathcal{S}_{M^\beta}$ on M^β such that

$$[\mathcal{Q}_K(M, \mathcal{S}, Z_\beta)]^K = \left[\mathcal{Q}_{K_\beta}(M^\beta, \mathcal{S}_{M^\beta} \otimes \mathrm{Sym}(\mathcal{N}), M^\beta \cap \Phi_{\mathcal{S}}^{-1}(\beta)) \otimes \bigwedge(\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}}\right]^{K_\beta}$$

Consequence

If the eigenvalues of $\frac{1}{i}\mathcal{L}(\beta)$ on $\mathcal{S}_{M^\beta} \otimes \bigwedge(\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}}$ are strictly positive then

$$[\mathcal{Q}_K(M, \mathcal{S}, Z_\beta)]^K = 0.$$

Function $d_{\mathcal{S}}$

Define $d_{\mathcal{S}} : Z_{\mathcal{S}} \longrightarrow \mathbb{R}$ by the following relation

$$d_{\mathcal{S}}(m) = \|\theta\|^2 + \frac{1}{2}\mathbf{nTr}_{\mathrm{T}_m M}|\theta| - \mathbf{nTr}_{\mathfrak{k}}|\theta|, \quad \text{with} \quad \theta = \Phi_{\mathcal{S}}(m).$$

where **nTr** is a **normalized** trace.

Facts

- $d_{\mathcal{S}}(m)$ is the smallest eigenvalue of $\frac{1}{i}\mathcal{L}(\theta)$ on $\mathcal{S}_{M^\theta}|_m \otimes \bigwedge(\mathfrak{k}/\mathfrak{k}_\theta)_{\mathbb{C}}$.
- $d_{\mathcal{S}}$ is a K-invariant locally constant function on $Z_{\mathcal{S}}$ that takes a finite number of values.

Localization on $Z_{\mathcal{S}}^{=0} := \{d_{\mathcal{S}} = 0\}$

If $d_{\mathcal{S}}$ is non-negative on M, we have

$$[\mathcal{Q}_K(M,\mathcal{S})]^K = \left[\mathcal{Q}_K(M,\mathcal{S},Z_{\mathcal{S}}^{=0})\right]^K$$

Shifting trick

By the shifting trick

$$m_{\mathcal{O}} = [\mathcal{Q}_K(M \times \mathcal{O}^*, \mathcal{S} \boxtimes \mathcal{S}_{\mathcal{O}^*})]^K$$

On the manifold $M \times \mathcal{O}^*$, we consider the set $Z_{\mathcal{O}}$ where the Kirwan vector field $\kappa_{\mathcal{S} \boxtimes \mathcal{S}_{\mathcal{O}^*}}$ vanishes, and the function

$$d_{\mathcal{O}} := d_{\mathcal{S} \boxtimes \mathcal{S}_{\mathcal{O}^*}} : Z_{\mathcal{O}} \to \mathbb{R}.$$

Theorem A

The function $d_{\mathcal{O}}$ takes non-negative values.

Corollary

We have

$$m_{\mathcal{O}} = \left[\mathcal{Q}_K(M \times \mathcal{O}^*, \mathcal{S} \boxtimes \mathcal{S}_{\mathcal{O}^*}, Z_{\mathcal{O}}^{=0})\right]^K.$$

Hence $m_{\mathcal{O}} = 0$ if $Z_{\mathcal{O}}^{=0} = \emptyset$.

Computation of $Z_{\mathcal{O}}^{=0}$

We have to understand when the subset $Z_{\mathcal{O}}^{=0} \subset M \times \mathcal{O}^*$ is non-empty.

Theorem B

- $Z_{\mathcal{O}}^{=0} \neq \emptyset$ only if $\exists\,(\mathfrak{h}) \in \mathcal{H}$ such that $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$.
- Suppose that $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ for some $(\mathfrak{h}) \in \mathcal{H}$. Then

$$Z_{\mathcal{O}}^{=0} = \coprod_{\mathcal{P}} Z_{\mathcal{O}}^{\mathcal{P}}$$

where the disjoint union is parametrized by $\mathcal{P} \in \mathcal{A}((\mathfrak{h})) \cap \Phi_{\mathcal{S}}(M)$ such that $\mathcal{Q}_K^{spin}(\mathcal{P}) = \pi_{\mathcal{O}}$.

Magical inequality

Let $T \subset K$ be a maximal torus with Lie algebra $\mathfrak{t}$.

The proofs of Theorems A and B use in a crucial way the following

Magical inequality

Let $\lambda, \mu \in \mathfrak{t}^*$, where λ is admissible and regular. We have

$$\|\lambda - \mu\|^2 \geq \|\rho(K_\mu)\|^2,$$

and the equality holds only if the following hold

• μ is admissible and belongs to the Weyl chamber defined by λ,

• $\mathcal{Q}_K^{spin}(K\mu) = \pi_{K\lambda}$.

Final computation

At this stage we know that

$$m_{\mathcal{O}} = \sum_{\mathcal{P}} \left[\mathcal{Q}_K(M \times \mathcal{O}^*, \mathcal{S} \boxtimes \mathcal{S}_{\mathcal{O}^*}, Z_{\mathcal{O}}^{\mathcal{P}})\right]^K$$

where the sum is parametrized by $\mathcal{P} \in \mathcal{A}((\mathfrak{h}))$ such that $\mathcal{Q}_K^{spin}(\mathcal{P}) = \pi_{\mathcal{O}}$.

With a bit more work we get

Final computation

$$\left[\mathcal{Q}_K(M \times \mathcal{O}^*, \mathcal{S} \boxtimes \mathcal{S}_{\mathcal{O}^*}, Z_{\mathcal{O}}^{\mathcal{P}})\right]^K = \mathcal{Q}^{spin}(M_{\mathcal{P}}).$$

THE END !