Real geometric quantisation

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- The original idea of geometric quantisation is to associate a Hilbert space to a symplectic manifold via a prequantum line bundle and a polarisation.

- I will show an approach to compute cohomology groups appearing in geometric quantisation of integrable systems with nondegenerate singularities.

The setup

- Let (M, ω) be a symplectic manifold such that $[\omega]$ is integral.

- (L, ∇^ω) a, Hermitian, complex line bundle with connection over M that satisfies $curv(\nabla^\omega) = -i\omega$.

- The symplectic manifold (M, ω) is called prequantisable and the pair (L, ∇^ω) a prequantum line bundle for (M, ω).

Polarisation

- A real polarisation $\mathcal{P}$ is an integrable (in the Sussmann sense) distribution of TM whose leaves are generically Lagrangian.

- An integrable system $F = (f_1, \ldots, f_n) : M^{2n} \to \mathbb{R}^n$ on a symplectic manifold defines a Lagrangian foliation (possibly with singularities): $\mathcal{P} = \langle X_{f_1}, ..., X_{f_n} \rangle_{C^\infty(M;\mathbb{R})}$.

- P will denote the complexification of $\mathcal{P}$: $P = \langle X_{f_1}, ..., X_{f_n} \rangle_{C^\infty(M;\mathbb{C})}$

Geometric quantisation (à la Kostant)

- $\mathcal{Q}(M) = \bigoplus_{k \geq 0} \check{H}^k(M; \mathcal{J})$ is the quantisation of $(M, \omega, L, \nabla, P)$.

- $\mathcal{J}$ is the sheaf of flat sections, i.e.: the space of local sections σ of L such that $\nabla^\omega_X \sigma = 0$ for any vector field X of P.

- $\nabla := \nabla^\omega \Big|_P$ is the restriction of the connection ∇^ω to the polarisation P.

Nondegenerate singularities

- A singular point of the moment map is called nondegenerate if the quadratic part of the first integrals form a cartan subalgebra.

- **Theorem:** *(Eliasson, Miranda and Zung) Near a compact singular nondegenerate orbit both the foliation and the symplectic form can be simultaneously linearised (in an equivariant way).*

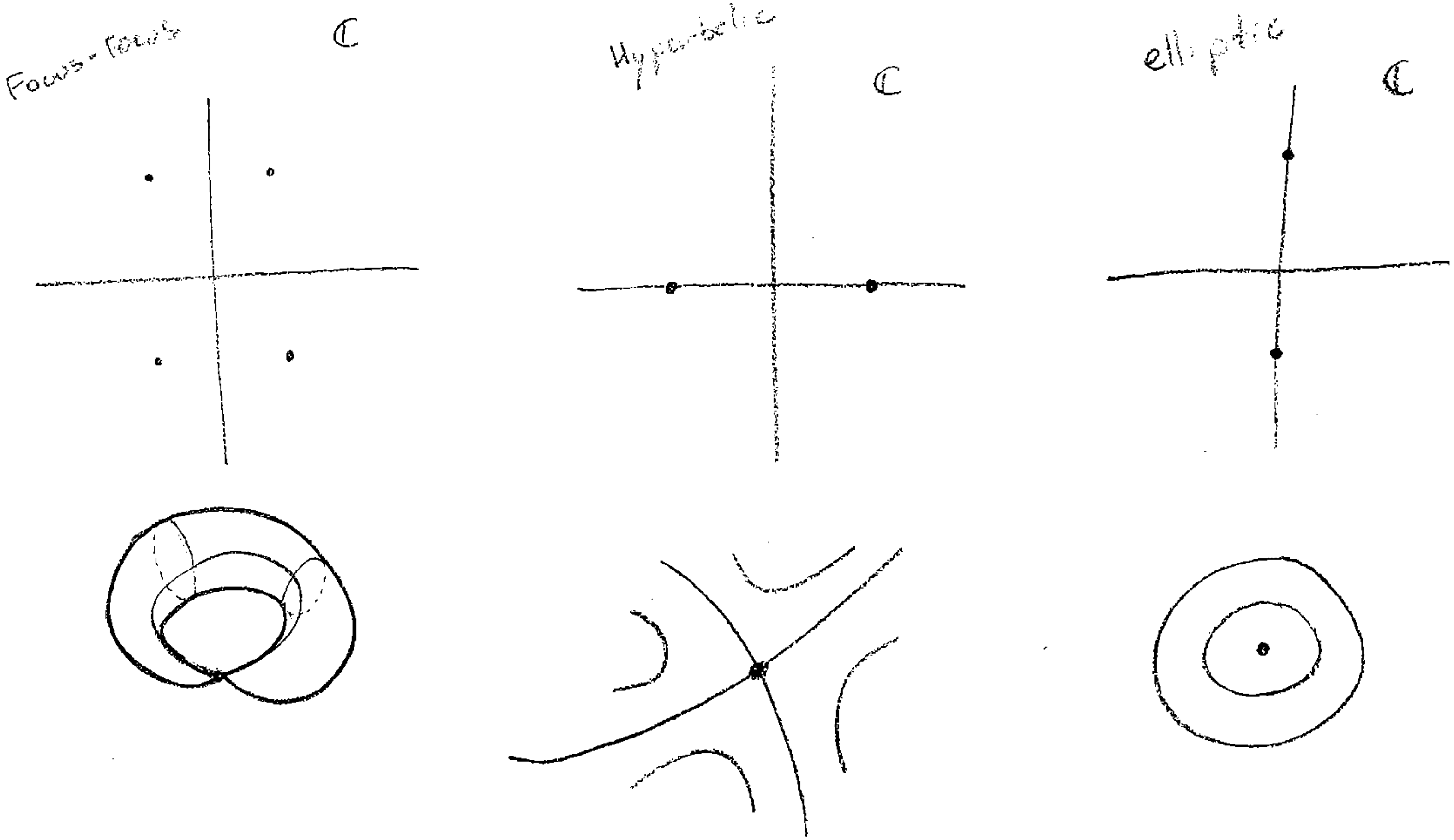
- The local model is given by $N = D^k \times \mathbb{T}^k \times D^{2(n-k)}$ and $\omega_0 = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i$.

- Regular $f_i = x_i$ for $i = 1, ..., k$;

- Elliptic $f_i = x_i^2 + y_i^2$ for $i = k+1, ..., k_e$;

- Hyperbolic $f_i = x_i y_i$ for $i = k_e + 1, ..., k_e + k_h$;

- Focus-focus $f_i = x_i y_{i+1} - x_{i+1} y_i$, $f_{i+1} = x_i y_i + x_{i+1} y_{i+1}$ for $i = k_e + k_h + 2j - 1$, $j = 1, ..., k_f$.


Focus-focus
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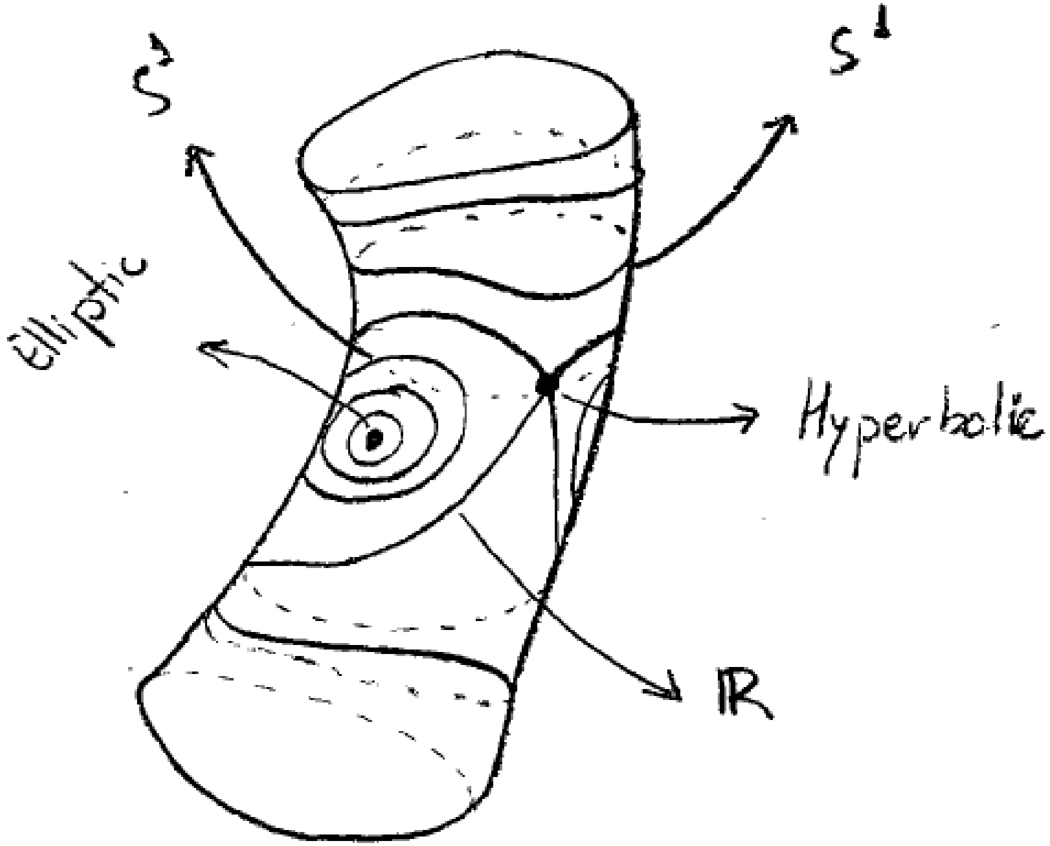
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Elliptic
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Resolution approach

- Let $\mathcal{S}_P^k(L)$ be the sheaf of $\Gamma(\bigwedge^k P^* \otimes_{C^\infty(M;\mathbb{C})} L)$ and d^∇ is the exterior derivative obtained by twisting d_P with the connection ∇.

- **Theorem:**

$$0 \longrightarrow \mathcal{J} \hookrightarrow \mathcal{S} \xrightarrow{\nabla} \mathcal{S}_P^1(L) \xrightarrow{\mathrm{d}^\nabla} \cdots \xrightarrow{\mathrm{d}^\nabla} \mathcal{S}_P^n(L) \xrightarrow{\mathrm{d}^\nabla} 0$$

is a fine resolution for $\mathcal{J}$. Therefore its cohomology computes geometric quantisation; $\check{H}^k(M;\mathcal{J}) \cong H^k(S_P{}^\bullet(L))$.

- Regular case: Kostant (Rawnsley).

- $(\mathcal{P}, [\cdot,\cdot]\big|_{\mathcal{P}})$ is a Lie subalgebra of $(\Gamma(TM), [\cdot,\cdot])$ and can be represented on $C^\infty(M)$ as vector fields acting on smooth functions.

$$0 \longrightarrow C^\infty_{\mathcal{P}}(M) \hookrightarrow C^\infty(M) \xrightarrow{\mathsf{d}_{\mathcal{P}}} \Omega^1_{\mathcal{P}}(M) \xrightarrow{\mathsf{d}_{\mathcal{P}}} \cdots \xrightarrow{\mathsf{d}_{\mathcal{P}}} \Omega^n_{\mathcal{P}}(M) \xrightarrow{\mathsf{d}_{\mathcal{P}}} 0$$

For $Y_1, \ldots, Y_{k+1} \in \mathcal{P}$, the coboundary operator is given by

$$\begin{aligned} \mathsf{d}_{\mathcal{P}}\alpha \quad (Y_1, \ldots, Y_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} Y_i(\alpha(Y_1, \ldots, \widehat{Y}_i, \ldots, Y_{k+1})) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([Y_i, Y_j], Y_1, \ldots, \widehat{Y}_i, \ldots, \widehat{Y}_j, \ldots, Y_{k+1}) \ . \end{aligned}$$

$C^\infty_{\mathcal{P}}(M) = \ker(\mathsf{d}_{\mathcal{P}})$

$\Omega^k_{\mathcal{P}}(M) = \mathrm{Hom}_{C^\infty(M)}(\bigwedge^k_{C^\infty(M)} \mathcal{P}; C^\infty(M))$
plus vanishing conditions

- **Theorem:** *(Miranda and Solha)*

  Consider $(\mathbb{R}^{2n}, \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i)$ *endowed with a real polarisation* $\mathcal{P}$ *with singularities of elliptic and hyperbolic type, then the foliated cohomology group in degree* 1 *is nontrivial — Poincaré lemma does not hold.*

- Almost toric case: Solha.

- Nondegenerate case (inclusion of hyperbolic singularities): Miranda and Solha.

Bohr-Sommerfeld condition

- A leaf ℓ of $\mathcal{P}$ is a Bohr-Sommerfeld leaf if there is a nonzero flat section $\sigma : \ell \to L$. Equivalently, all loops in ℓ have trivial holonomy.

- **Theorem:** *Under the assumption that the zero fibre is Bohr-Sommerfeld, the image of Bohr-Sommerfeld fibres by a moment map is contained in $\mathbb{R}^{n-k} \times \mathbb{Z}^k$; k being the number of linearly independent Hamiltonian S^1-actions generated by the moment map.*

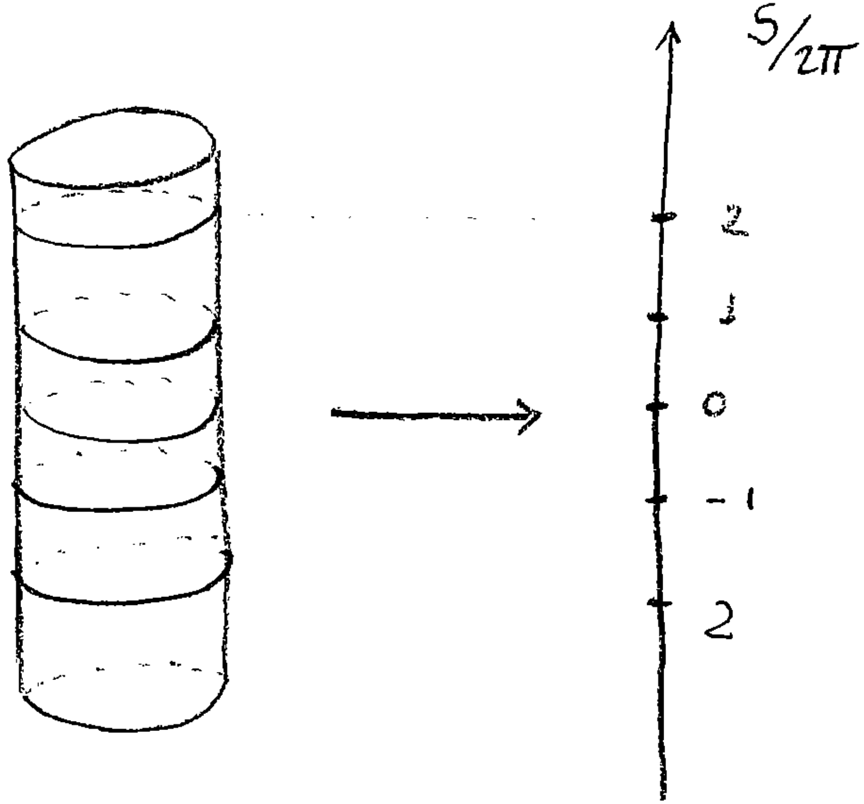
- The Liouville tori case was proved by Guillemin and Sternberg: their theorem holds for Lagrangian fibrations, with compact connected fibres, over simply connected basis.

- Over a Bohr-Sommerfeld fibre each component of the moment map generating a S^1-action takes an integer value, depending only on the fibre.

  Lemma: *Let X be the generator of a symplectic S^1-action with orbits γ, then its (2π) holonomy can be computed (up to flat bundles) by*

$$hol_{\nabla^\omega}(\gamma) = \mathrm{e}^{i2\pi\theta(X)} \;;$$

  where θ is a particular invariant potential 1-form for ω in a neighbourhood of γ.


S/2π
2
1
0
-1
2

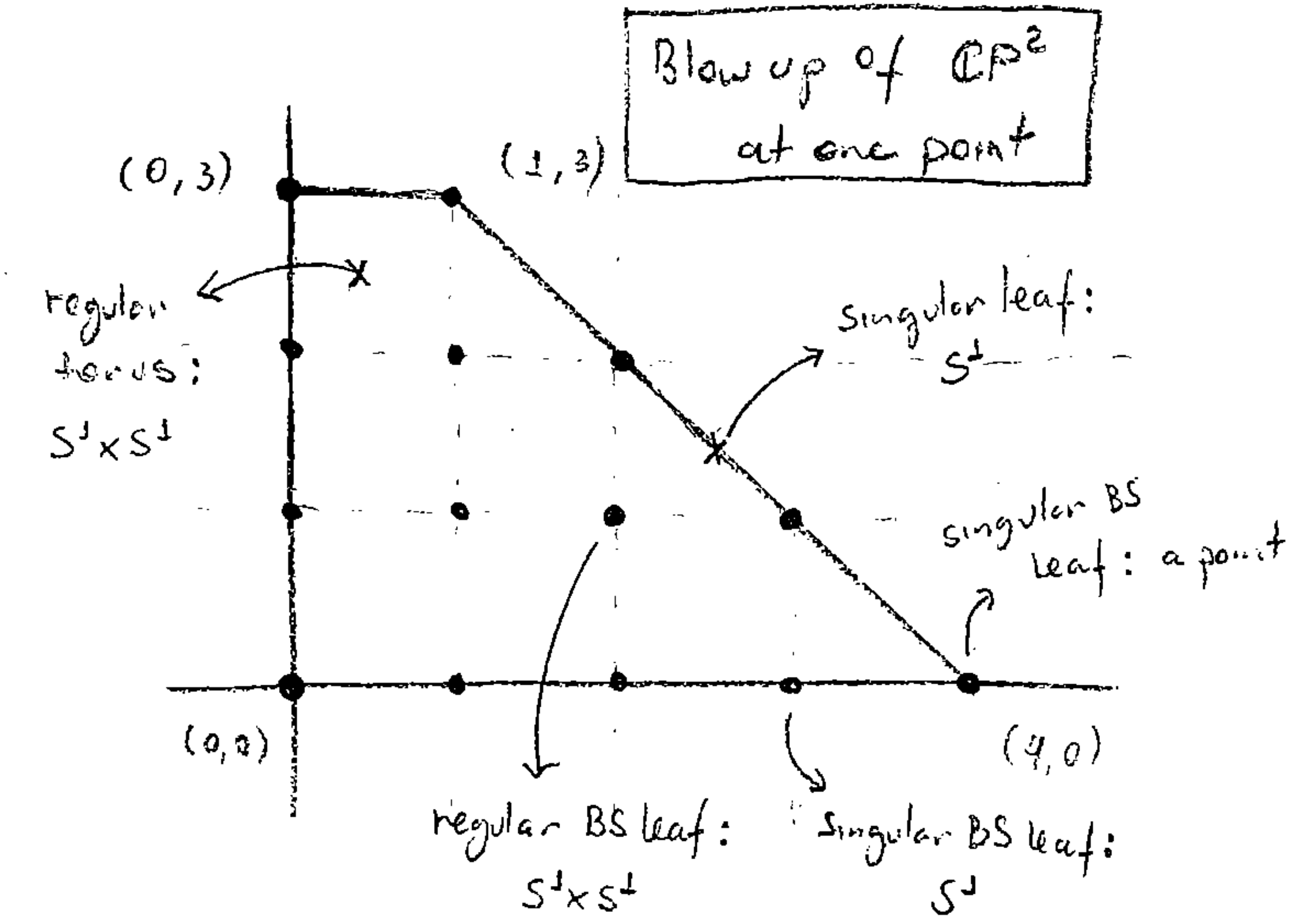
Nonsingular and elliptic cases

- **Theorem:** *(Sniatycki) If the base space N is a manifold and the natural projection $\mathcal{F}: M \to N$ is a Lagrangian fibration, then* $\mathcal{Q}(M) = \check{H}^k(M;\mathcal{J}) \cong \check{H}^0(\ell_{BS};\mathcal{J}\big|_{\ell_{BS}})$.

  k is the rank of the fundamental group of a fibre and $\ell_{BS} \subset M$ is the union of all Bohr-Sommerfeld fibres.

- **Theorem:** *(Hamilton) For a $2n$-dimensional compact locally toric fibration:*

$$\mathcal{Q}(M) = \check{H}^n(M;\mathcal{J}) \cong \bigoplus_{p \in BS_r} \mathbb{C} \ .$$



Blow up of $\mathbb{CP}^2$ at one point
(0, 3)
(1, 3)
regular torus: $S^1 \times S^1$
singular leaf: S^1
singular BS leaf: a point
(0, 0)
(4, 0)
regular BS leaf: $S^1 \times S^1$
Singular BS leaf: S^1

- Sniatycki's result:

  1. trivially holds without metaplectic correction and does not need compactness assumptions;

  2. depends on the existence of global action-angle coordinates and the image of the moment map must be a manifold.

- Hamilton's result:

  1. can be adapted, however does not include metaplectic correction, and needs compactness of the fibres;

  2. does not require a manifold structure on the image of the moment map.

- For the new proofs:

  1. it does not matter if metaplectic correction is included or not (it holds for Hermitian line bundles with flat connection along the leaves of the polarisation);

  2. it does not need a manifold structure on the image of the moment map;

  3. Sniatycki's and Hamilton's result are unified;

  4. something can be said about the almost toric case.

Dimension 2

- **Theorem:** *(Hamilton and Miranda) For a integrable system on a compact surface, whose moment map has only nondegenerate singularities*

$$\mathcal{Q}(M) = \check{H}^1(M;\mathcal{J}) \cong \bigoplus_{p\in\mathcal{H}} (\mathbb{C}^{\mathbb{N}} \oplus \mathbb{C}^{\mathbb{N}}) \oplus \bigoplus_{p\in BS_r} \mathbb{C} \ .$$

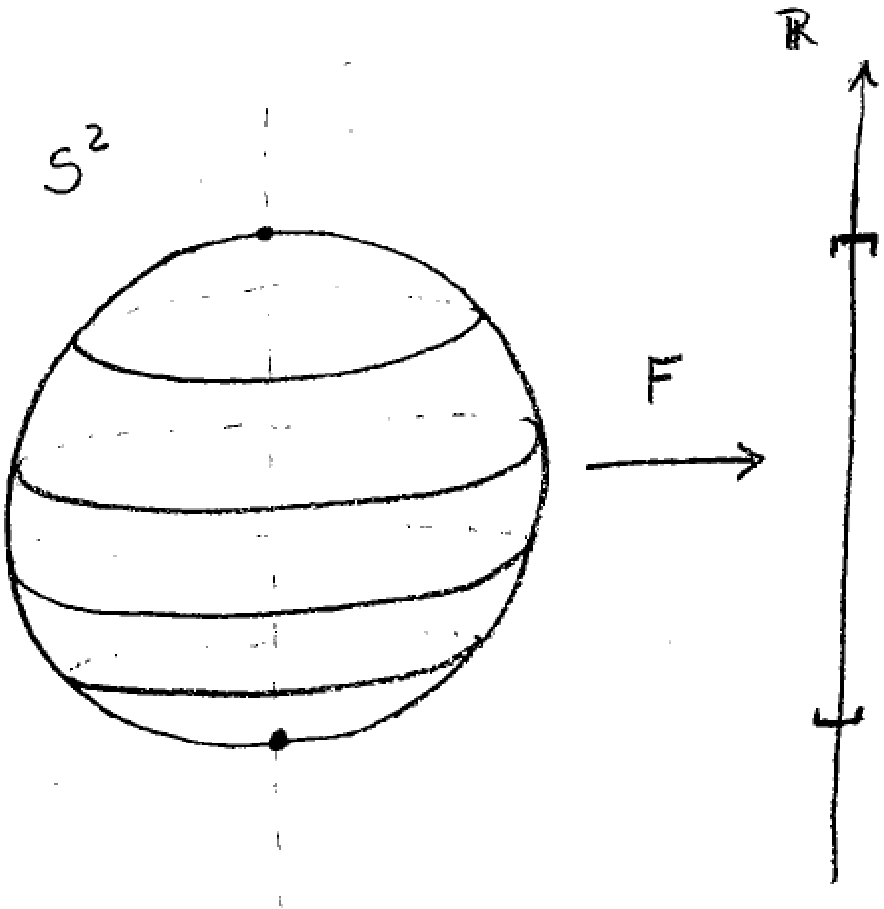
Free rigid body

- The coadjoint orbits of $\mathfrak{so}(3)^*$ are spheres, with the area form, and a point.

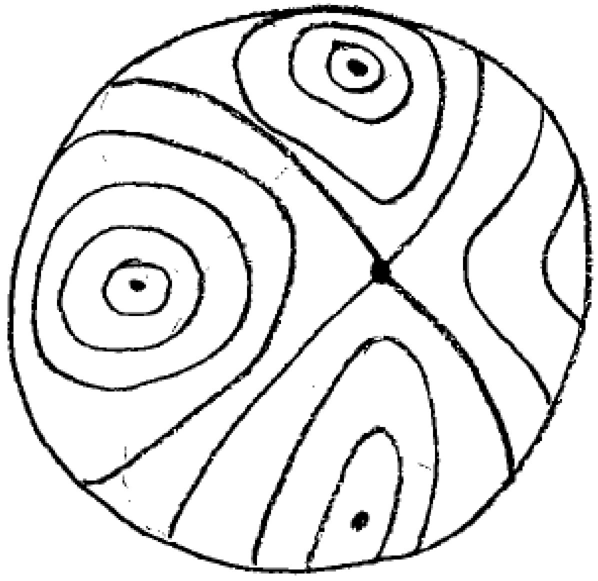
- The Euler equations are equivalent to the free rigid body dynamics.

- If the principal axis of inertia satisfies $I_1 = I_2 \neq I_3$, the momentum of inertia operator has 2 distinct eigenvalues. It has two elliptic singularities.

- A generic body satisfies $I_1 < I_2 < I_3$. It has four elliptic and two hyperbolic singularities.



S^2
F
$\mathbb{R}$



Rawnsley: Circle actions and homotopy operators

$$0 \longrightarrow \mathcal{J} \hookrightarrow \mathcal{S} \xrightarrow{\nabla} \mathcal{S}_P^1(L) \xrightarrow{\mathrm{d}^\nabla} \cdots \xrightarrow{\mathrm{d}^\nabla} \mathcal{S}_P^n(L) \xrightarrow{\mathrm{d}^\nabla} 0$$

- Suppose that $X \in \mathcal{P}$ is a periodic Hamiltonian vector field and ϕ_t represents its flow.

- $$\boldsymbol{J}_X(\alpha \otimes s) = \int_0^{2\pi} (\imath_X \circ \phi_t^* \alpha) \otimes \Pi_{\phi_t}^{-1}(s \circ \phi_t) \, \mathrm{d}t$$

  resembles an homotopy operator:

  $$[hol_{\nabla^\omega}(\gamma)^{-1} - 1]\alpha \otimes s = \boldsymbol{J}_X(\mathrm{d}^\nabla \alpha \otimes s) + \mathrm{d}^\nabla \boldsymbol{J}_X(\alpha \otimes s) \ .$$

- For sections of L,

$$[hol_{\nabla^\omega}(\gamma)^{-1} - 1]s = \boldsymbol{J}_X(\nabla s)$$

and if the holonomy is nontrivial over a dense set $H^0(S_P{}^\bullet(L)) = \{0\}$.

- If $\boldsymbol{\alpha} \in S_P^k(L)$ is closed, $\mathrm{d}^\nabla \boldsymbol{\alpha} = 0$,

$$[hol_{\nabla^\omega}(\gamma)^{-1} - 1]\boldsymbol{\alpha} = \mathrm{d}^\nabla \boldsymbol{J}_X(\boldsymbol{\alpha}) \ ,$$

and if the holonomy is nontrivial over a dense set $\boldsymbol{J}_X(\boldsymbol{\alpha})$ closed $\Rightarrow H^k(S_P{}^\bullet(L)) = \{0\}$.

Proposition: *Supposing that* (M, ω) *admits a symplectic* S^1*-action preserving* P*,* $H^0(S_P{}^\bullet(L)) = \{0\}$ *if* $\{hol_{\nabla^\omega}(\gamma) \neq 1\}$ *is dense.*

Proposition: *Let* $\boldsymbol{\alpha} \in S_P^k(L)$ *be closed,* $\mathrm{d}^\nabla \boldsymbol{\alpha} = 0$*, and* $k \neq 0$*.*

- *The form* $\boldsymbol{\alpha}$ *is exact everywhere* $hol_{\nabla^\omega}(\gamma) \neq 1$*. It is also globally exact if* $\{hol_{\nabla^\omega}(\gamma) \neq 1\}$ *is dense and* $\boldsymbol{J}_X(\boldsymbol{\alpha}) = 0$ *where* $hol_{\nabla^\omega}(\gamma) = 1$*.*

- *When* $\{hol_{\nabla^\omega}(\gamma) = 1\}$ *is a (not necessarily connected) submanifold,* $\boldsymbol{\alpha}$ *is exact on* M *if and only if* $\boldsymbol{J}_X(\boldsymbol{\alpha})\Big|_{T\{hol_{\nabla^\omega}(\gamma)=1\}}$ *is exact.*

Computing the building blocks: regular case

- The existence of $\boldsymbol{J}_X$ implies no global flat sections and induces an isomorphism between flat sections over Bohr-Sommerfeld leaves and the first cohomology group.

$$\Psi : S^1_P(L) \to \bigoplus_{k\in\mathbb{Z}} \Gamma(L|_{\ell_k}) \quad \Psi(\boldsymbol{\alpha}) = \oplus_{k\in\mathbb{Z}} \boldsymbol{J}_X(\boldsymbol{\alpha})\Big|_{\ell_k} .$$

Proposition: *The quantisation of a cylinder polarised by circles is $\mathbb{C}^{b_s}$, where b_s is the number of Bohr-Sommerfeld leaves.*

- Elliptic singularities give no contribution (Poincaré lemma).

  1. $hol_{\nabla^\omega}(\gamma) = \mathrm{e}^{i2\pi\theta(X)} = \mathrm{e}^{i2\pi(x^2+y^2)}$ and holonomy is non-trivial over a dense set: $H^0(S_P{}^\bullet(L)) = 0$.

  2. Since the origin is a fixed point, the operator $\boldsymbol{J}_X$ is the null operator when restricted to the origin (trivial holonomy).

  3. Hence for each contractible neighbourhood of the origin that does not contain any other Bohr-Sommerfeld leaf, elliptic singularities give no contribution to quantisation (first proved by Hamilton using different techniques).

     Proposition: *The quantisation of an open disk polarised by circles is $\mathbb{C}^{b_s}$, where b_s is the number of nonsingular Bohr-Sommerfeld leaves.*

- Focus-focus:

1. **Theorem:** *(Zung) There exists a neighbourhood of a focus-focus singular fibre over which a Hamiltonian S^1-action is defined and its vector field is one of the generators of the polarisation given by the nondegenerate integrable system.*

2. Poincaré lemma: In a small enough neighbourhood W of a singular point of a focus-focus Bohr-Sommerfeld fibre $\boldsymbol{J}_X$ is the null operator over the points where $\{hol_{\nabla^\omega}(\gamma) = 1\}$.

Proposition: *In the neighbourhood of* ℓ_{ff} *over which a Hamiltonian* S^1*-action is defined there exists a neighbourhood* V *containing only* ℓ_{ff} *as a Bohr-Sommerfeld fibre such that* $\check{H}^0(V;\mathcal{J}\big|_V) = \{0\}$.

- The first, nonlocal, obstacle is that $\{hol_{\nabla^\omega}(\gamma) = 1\}$ is not a submanifold, and one needs to prove that $\boldsymbol{J}_X$ is the null operator over the points where $\{hol_{\nabla^\omega}(\gamma) = 1\}$.

- Another approach would be to prove only the exactness of $\boldsymbol{J}_X$ and check out convergence over the singular points of $\{hol_{\nabla^\omega}(\gamma) = 1\}$.

- Hyperbolic:

  1. **Theorem 1:** *(Miranda and Solha)*

     All cohomology groups vanish near a singularity of purely hyperbolic type in dimension 2 and 4.

  For $X(g) = ifg + \alpha(X)$ a solution is given by $g = \tilde{g} - G$, where $\tilde{g}$ is obtained as a formal solution via Taylor series, $X(\tilde{g}) = if\tilde{g} + \alpha(X) + T$, and $G = \int_{-\ln\gamma}^{0} \mathrm{e}^{-ift} T \circ \phi_t \, \mathrm{d}t$ is a solution for flat functions: $X(G) = ifG + T$.

  For higher degrees, one has to solve a system of equations of this type together with compatibility conditions (that are not only necessary, but sufficient).

Künneth formula

- Let $((-1,1)\times S^1, \mathrm{d}x\wedge \mathrm{d}y, \langle\frac{\partial}{\partial y}\rangle)$ and $(M,\omega=\mathrm{d}\theta, L,\nabla, P)$ arbitrary.

- $\mathcal{L}\to((-1,1)\times S^1\times M, \mathrm{d}x\wedge\mathrm{d}y+\omega)$ is a prequantum line bundle.

  Theorem:

  The map $\Psi: H^k(S_{\langle\frac{\partial}{\partial y}\rangle\oplus P}{}^\bullet(\mathscr{L}))\to H^{k-1}(S_P{}^\bullet(L))$ *defined by* $\Psi([\bar{\alpha}])=\left[\boldsymbol{J}_{\frac{\partial}{\partial y}}(\bar{\boldsymbol{\alpha}})\Big|_{\{hol_{\nabla^\omega}(\gamma)=1\}}\right]$ *is an isomorphism.*

- If the first factor, N, is an elliptic or focus-focus local model, $\boldsymbol{J}_X$ implies that $\mathcal{Q}(N \times M) = \{0\}$ for any M (exact, $\omega = \mathrm{d}\theta$).

- The Hamiltonian S^1-action has nontrivial holonomy over a dense set, and the set $\{hol_{\nabla^\omega}(\gamma) = 1\}$ are fixed points: the operator $\boldsymbol{J}_X$ is the null operator when restricted to $\{hol_{\nabla^\omega}(\gamma) = 1\}$.