Part 0. Motivation

In the paper arXiv:1107.1741, 'K-homology and index theory on contact manifolds', Paul F. Baum and Erik van Erp showed how to construct an index problem from a sub-elliptic operator using Kasparov theory. (I will explain in the first part of this talk some aspects of Kasparov theory).

In arXiv:1107.0805 'Index theory for locally compact noncommutative geometries', my co-authors and I obtained the local index formula for non-unital algebras in a very general setting without using compact support type assumptions by exploiting the Kasparovian viewpoint.

In arXiv:1004.1582 'The index formula and the spectral shift function for relatively trace class perturbations', the authors Fritz Gesztesy, Yuri Latushkin, Konstantin A. Makarov, Fedor Sukochev, Yuri Tomilov extended work of Robbin-Salaman on the relationship between spectral flow and the Fredholm index by incorporating some ideas of Azamov (from his thesis) and some permutations of Dodds, Sukochev and myself. At this point Kasparov does not enter.

In arXiv:1110.1472,. 'Spectral flow and the unbounded Kasparov product', by Jens Kaad and Matthias Lesch looked at a situation where Kasparov theory was brought to bear on a generalisation of the problem in the previous article.

So I became interested in whether there was a way to generalise these ideas to study a more general picture of spectral flow. In this approach we start with the question of whether there is a notion of spectral flow between operators that are not Fredholm but which individually represent unbounded Kasparov classes.

Evidence that this is not completely wrong is that the relation between spectral flow and the spectral shift function is not exact. The latter exists in more generality than the former. The question then is whether Kasparov theory can be used to explain why the spectral shift function can legitmately be regarded as giving spectral flow when non-Fredholm operators are involved.

This is not an unreasonable strategy given the work in the previously listed arXiv papers. We are also interested in whether the so-called Witten index can be brought into the picture.

Next: explain some of the terminology!

Part I: Kasparov picture.
$(\mathcal{N}, \tau)$ denotes a semifinite von Neumann algebra with τ a semifinite normal faithful trace. $\mathcal{K}_{\mathcal{N}}$ is the ideal of τ-compact operators in $\mathcal{N}$.

A nonunital Fréchet sub-$*$-algebra $\mathcal{A}$ of $\mathcal{N}$ is called a *pre-C^*-algebra* if it is stable under the holomorphic functional calculus. (This means that its minimal unitalization $\mathcal{A}^{\sim} := \mathcal{A} \oplus \mathbf{C}$ is stable under the (ordinary) holomorphic functional calculus in the minimal unitalization of its C^*-completion.)

Definition. A **nonunital semifinite spectral triple** $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, relative to $(\mathcal{N}, \tau)$, is given by a Hilbert space $\mathcal{H}$, a pre-C^*-algebra $\mathcal{A} \subset \mathcal{N}$ acting on $\mathcal{H}$, and a densely defined unbounded self-adjoint operator $\mathcal{D}$ affiliated to $\mathcal{N}$ such that
1) $da := [\mathcal{D}, a]$ extends to a bounded operator in $\mathcal{N}$ for all $a \in \mathcal{A}$,
2) $a(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{K}(\mathcal{N}, \tau)$ for all $a \in \mathcal{A}$

$(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is even if there is a $\mathbf{Z_2}$-grading such that $\mathcal{A}$ is even and $\mathcal{D}$ is odd. Otherwise we say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is odd.

Note. 'Nonunital' here means that $(1+\mathcal{D}^2)^{-1}$ is not compact **and** that if $\mathcal{A}$ has a unit it is not the unit of $\mathcal{N}$.

The Kasparov class of a spectral triple
Essentially known from Kaad, Nest and Rennie (2008) (related to older results of Connes-Cuntz (1987)).

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}$ be a nonunital semifinite spectral triple relative to $(\mathcal{N}, \tau)$. Set $F_{\mathcal{D}} = \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$. Then for all $a \in \mathcal{A}$ and $\varphi \in C_0(\mathbf{R})$, $[F_{\mathcal{D}}, a]$, $\quad a\varphi(\mathcal{D})$. are τ-compact.

$\mathcal{K}_{\mathcal{N}}$ as a right $\mathcal{K}_{\mathcal{N}}$ C^*-module via $(b_1|b_2) := b_1^* b_2$, left multiplication by $F_{\mathcal{D}}$ on $\mathcal{K}_{\mathcal{N}}$ gives $F_{\mathcal{D}} \in End_{\mathcal{K}_{\mathcal{N}}}(\mathcal{K}_{\mathcal{N}})$.

Left multiplication by $a \in A$, the C^*-completion of $\mathcal{A}$, gives a representation of A as adjointable endomorphisms of $\mathcal{K}_{\mathcal{N}}$

$[F_{\mathcal{D}}, a] \in \mathcal{K}_{\mathcal{N}} = End^0_{\mathcal{K}_{\mathcal{N}}}(\mathcal{K}_{\mathcal{N}})$, the compact endomorphisms, for all $a \in A$.

Since $a(F_{\mathcal{D}}^2 - 1) \in \mathcal{K}_{\mathcal{N}}$ and $F_{\mathcal{D}} = F_{\mathcal{D}}^*$ by construction, we obtain a Kasparov module $({}_A(\mathcal{K}_{\mathcal{N}})_{\mathcal{K}_{\mathcal{N}}}, F_{\mathcal{D}})$ with class $[(\mathcal{K}_{\mathcal{N}}, F_{\mathcal{D}})] \in KK^j(A, \mathcal{K}_{\mathcal{N}})$, where j is 0 iff our spectral triple was $\mathbf{Z_2}$-graded.
Using the Kasparov product we now have a well-defined map

$$\cdot \otimes_A [(\mathcal{K}_{\mathcal{N}}, F_{\mathcal{D}})] \,:\, K_j(A) \to K_0(\mathcal{K}_{\mathcal{N}})$$

Write X for the Kasparov module $({}_A(\mathcal{K}_{\mathcal{N}})_{\mathcal{K}_{\mathcal{N}}}, F_{\mathcal{D}})$. Assume we choose a representative of the class of this Kasparov module with $F^2 = 1$.

Represent elements $a + \lambda Id_{A^\sim}$ on X as $a + \lambda Id_X$, $\lambda \in \mathbf{C}$.

Even case:
Suppose that e and f are projections in a (matrix algebra over the minimal unitization $A^\sim$) and suppose also that we have a class $[e] - [f] \in K_0(A)$.

Let $F_+ = \frac{1}{4}(1-\gamma)F(1+\gamma)$, then $eF_+e : e\frac{1+\gamma}{2}X \to e\frac{1-\gamma}{2}X$ is Fredholm,

$$[\text{Index}(eF_+e)] - [\text{Index}(fF_+f)]$$

$$= [\ker eF_+e] - [\operatorname{coker} eF_+e] - [\ker fF_+f] + [\operatorname{coker} fF_+f],$$

and the individual terms are the classes of finite projective $\mathcal{K}_\mathcal{N}$ modules.

The odd case:

$$[u] \otimes_A [(X,F)] = [\text{Index}(\frac{1+F}{2}u\frac{1+F}{2} - \frac{1-F}{2})] \in K_0(\mathcal{K}_\mathcal{N}),$$

where $[u] \in K_1(A)$. Writing $(1+F)/2 = P$ for the positive spectral projection of F, we have

$$[u] \otimes_A [(X,F)] = [\text{Index}(PuP)] \in K_0(\mathcal{K}_\mathcal{N}),$$

and both $\ker PuP$ and cokerPuP are finite projective $\mathcal{K}_\mathcal{N}$-modules.

Problem For non-type I semifinite von Neumann algebras the compacts are not countably generated (a fundamental assumption for Kasparov modules).

The solution is to restrict to separable algebras $\mathcal{A}$ and replace the compacts by a countably generated subalgebra constructed from commutators of $\mathcal{A}$ with $F_{\mathcal{D}}$.

Specifically we need all operators of the form

$$[F_D, a], F_{\mathcal{D}}[F_{\mathcal{D}}, a], b[F_{\mathcal{D}}, a], F_{\mathcal{D}} b[F_{\mathcal{D}}, a], a\phi(\mathcal{D})$$

where ϕ is continuous of compact support. These generate an algebra which forms a bimodule of the right kind.

Fredholm modules

A semifinite **Fredholm module** for a $*$-algebra $\mathcal{A}$ relative to $(\mathcal{N}, \tau)$ is a pair $(\mathcal{H}, F)$ where $\mathcal{A}$ is (continuously) represented in $\mathcal{N}$ and F is a self-adjoint operator in $\mathcal{N}$ satisfying:

1. $a(1 - F^2) \in \mathcal{K}_{\mathcal{N}},\ and$ 2. $[F, a] \in \mathcal{K}_{\mathcal{N}}\ for\ a \in \mathcal{A}$.

Summability:

if $[F, a] \in \mathcal{L}^{p+1}(\mathcal{N}, \tau)$ for $a \in \mathcal{A}$, we say that $(\mathcal{H}, F)$ is $p+1$-summable. The spectral dimension of such a module is the infimum of those n such that $[F, a] \in \mathcal{L}^n(\mathcal{N}, \tau)$ for all $a \in \mathcal{A}$.

In the spectral triple version this is the requirement $a(1+\mathcal{D}^2)^{-s/2} \in \mathcal{L}^1(\mathcal{N}, \tau)$ for all $s > p$ and p (the spectral dimension) is the infimum of such s.

Lemma. Given a semifinite finitely summable spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, with spectral dimension p, then setting $F_{\mathcal{D}} := \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}$ yields, a semifinite $\lfloor p \rfloor + 1$-summable Fredholm module for $\mathcal{A}$.

Given a Fredholm module $(\mathcal{H}, F)$ relative to $(\mathcal{N}, \tau)$ we have a Kasparov module $(\mathcal{K}_{\mathcal{N}}, F_{\mathcal{D}})$ and the following diagram commutes

$$\begin{array}{ccc} (\mathcal{A}, \mathcal{H}, \mathcal{D}) & \to & (\mathcal{K}_{\mathcal{N}}, F_{\mathcal{D}}) \\ \downarrow & \nearrow & \\ (\mathcal{H}, F) & & \end{array} .$$

Next: a trick due to Alain Connes gives a representative of this class with $F^2 = 1$.

For any $\mu > 0$, define the 'double' of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ to be $(\mathcal{A}, \mathcal{H}^2, \mathcal{D}_\mu, M_2(\mathcal{N}), \tau \otimes \mathrm{Tr}_2)$, with $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$ and the action of $\mathcal{A}$ and $\mathcal{D}_\mu$ given by

$$\mathcal{D}_\mu := \begin{pmatrix} \mathcal{D} & \mu \\ \mu & -\mathcal{D} \end{pmatrix}, \qquad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad \forall a \in \mathcal{A}.$$

If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is even and graded by γ then the double is even and graded by $\gamma \oplus -\gamma$. $\mathcal{D}_\mu$ is invertible, and $F_\mu = \mathcal{D}_\mu |\mathcal{D}_\mu|^{-1}$ has square 1.

Lemma. The KK-classes associated with $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and $(\mathcal{A}, \mathcal{H}^2, \mathcal{D}_\mu)$ coincide. A representative of this class is $(\mathcal{K}^2_\mathcal{N}, F_\mu)$ with $F_\mu = \mathcal{D}_\mu |\mathcal{D}_\mu|^{-1}$.

The spectral dimension p is the same for the double.

The $K_0(\mathcal{K}_\mathcal{N})$-valued index pairings defined by the two spectral triples and the semifinite Fredholm module all agree: for $x \in K_*(\mathcal{A})$ of the appropriate parity and $\mu > 0$,

$$x \otimes_A [(\mathcal{A}, \mathcal{H}, \mathcal{D})] = x \otimes_A \left[\left(\mathcal{A}, \mathcal{H}^2, \mathcal{D}_\mu, M_2(\mathcal{N}), \tau \otimes \mathrm{Tr}_2\right)\right]$$

$$= x \otimes_A [(\mathcal{K}^2_\mathcal{N}, F_\mu)] \, .$$

The τ-finite operators $\mathcal{F}_\mathcal{N} \subset \mathcal{K}_\mathcal{N}$ are stable under the holomorphic functional calculus, and so $K_0(\mathcal{K}_\mathcal{N}) = K_0(\mathcal{F}_\mathcal{N})$. Thus we can always represent elements of $K_0(\mathcal{K}_\mathcal{N})$ by classes $[e] - [f]$ with $e, f \in \mathcal{F}^\sim_\mathcal{N}$ where $\sim$ denotes the one-point unitization. The trace τ defines a homomorphism $\tau_* : K_0(\mathcal{K}_\mathcal{N}) \to \mathbf{R}$ and the numerical index from the Fredholm module.

So we may define a numerical index pairing from a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ as follows:

1) Take the Kasparov product with the KK-class defined by the double spectral triple

$$\cdot \otimes_A [(\mathcal{K}_{\mathcal{N}}^2, F_\mu)] \, : \, K_j(A) \to K_0(\mathcal{K}_{\mathcal{N}}),$$

2) Apply the homomorphism $\tau_* : K_0(\mathcal{K}_{\mathcal{N}}) \to \mathbf{R}$ to the resulting class.

We will denote this pairing by

$$\langle x, (\mathcal{A}, \mathcal{H}, \mathcal{D}) \rangle \in \mathbf{R}, \quad x \in K_j(\mathcal{A}).$$

Let $u \in M_n(\mathcal{A}^{\sim})$ be a unitary. We remark firstly that, since $\mathcal{D}_\mu$ is invertible, the spectral flow from $\mathcal{D}_\mu \otimes \mathbf{1}_n$ to $\hat{u}(\mathcal{D}_\mu \otimes \mathbf{1}_n)\hat{u}^*$, written abusively $sf(\mathcal{D}, u\mathcal{D}u^*)$, is well defined.

Here

$$u \mapsto \hat{u} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix},$$

in the double.

This is because $\mathcal{D}_\mu$ is Fredholm (as it is invertible) and then by work of Phillips et al $sf(\mathcal{D}, u\mathcal{D}u^*)$ is the index of $P_\mu \hat{u} P_\mu$ which may be shown to be the index of PuP where $P_\mu = \chi_{[0,\infty)}(\mathcal{D}_\mu)$, $P = \chi_{[0,\infty)}(\mathcal{D})$,

It seems this double trick is not such a good plan when we want to study the Witten index using Kasparov modules. (NB the idea that this can be done is at the moment just a conjecture).

Conjectured approach

We are interested in defining spectral flow between unbounded operators $\mathcal{D}$ and $\mathcal{D} + A$ where A is bounded and $A(1 + \mathcal{D}^2)^{-1/2}$ is compact. This is compatible with the Kasparov module picture. Note that we use the straight line path $\mathcal{D} + tA$ for $t \in [0, 1]$.

Now we do not know that either $F_{\mathcal{D}}$ or $F_{\mathcal{D}+A}$ are Fredholm but we do know that their difference is compact appealing to a lemma from our previous work (all this is evolving from discussions between myself, Gayral, Phillips, Rennie and Sukochev).

So they are equal in the Calkin algebra $\mathcal{N}/\mathcal{K}_\tau$.

Question. Let $1 = P_0 + P^> + P^<$ be the decomposition of the identity into positve negative and kernel projections for $F_{\mathcal{D}}$. Similarly $1 = Q_0 + Q^> + Q^<$ for $F_{\mathcal{D}+A}$. Then are there checkable conditions such that $P_0, +P^>$ $Q_0 + Q^>$ are norm close when projected into the Calkin algebra?

Following earlier work of Phillips we would define spectral for the path above as the index of PQ as an operator from $Q\mathcal{H}$ tp $P\mathcal{H}$ where $P = P_0 + P^>$ and $Q = Q_0 + Q^>$.

Problem is to justify this definition in terms of what is normally understood about spectral flow.

(i) Everything is consistent with the case of ‘unitarily equivalent endpoints’.

Potential additional checks

Can we extend existing formulas for spectral flow to this situation?

For example find a formula in terms of the spectral shift function.

Does it help with the Witten index problem in the non-Fredholm case?

The ‘problem’ is that the Witten index is not an integer in this case.

Note that the Witten index arises in the even case of Kasparov modules and one passes from the spectral flow problem to the Witten index by a kind of suspension process.

However the Witten index question is known to be problematic from work on APS problems on a cylinder.

In fact Lesch has suggested that the Witten index is an ‘interior’ contribution to the true Fredholm index and that there is a contribution from eta type terms on the boundary at infinity on the cylinder.