Dualities in Field Theories and the Role of K-Theory

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Plan of the Lectures

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2. K-Theory and its Relevance to Physics
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5. The Topology of T-Duality and the Bunke-Schick Construction
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Part I

The Basic Idea of Dualities in Field Theories and T-Duality

A Few Key Developments in the History of Mathematical Physics

- Fermat's Theory of Optics (1662)
- Newton's Theory of Gravitation (*Principia mathematica*, 1687)
- Lagrange's Theory of Mechanics (*Mécanique analytique*, 1788)
- Hamilton's Theory of Mechanics (ca. 1835)
- Maxwell's Equations of Electromagnetism (ca. 1861)
- Birth of Quantum Mechanics (ca. 1925)
- Quantum Electrodynamics (QED, ca. 1948)
- Yang-Mills Theory (1954)
- Quantum Chromodynamics (QCD, ca. 1965–1975)
- Superstring Theory (ca. 1980–)

Fields

Most physical theories describe fields[1], e.g., the *gravitational field*, *electric field*, *magnetic field*, etc. Fields can be

- scalar-valued functions (scalars),
- sections of vector bundles (vectors),
- connections on principal bundles (special cases of gauge fields),
- sections of spinor bundles (spinors).

[1] In French, resp. German, *champs*, resp. *Feld*, not *corps* or *Körper*.

Lagrangians and Least Action

In classical physics, the fields satisfy a variational principle — they are critical points of the action S, which in turn is the integral of a local functional $\mathcal{L}$ called the Lagrangian. This is called the *principle of least action*. The Euler-Lagrange equations for critical points of the action are the equations of motion.

Examples

Let M be a 4-manifold, say compact.

1. Yang-Mills Theory. Field is a connection A on a principal G-bundle. "Field strength" F is the curvature, a $\mathfrak{g}$-valued 2-form. Action is $S = \int_M \operatorname{Tr} F \wedge *F$.
2. General Relativity (in Euclidean signature). Field is a Riemannian metric g on M. Action is $S = \int_M R \, d\text{vol}$, $R =$ scalar curvature. Field equation is Einstein's equation.

Lorentz *vs.* Euclidean Signature

One point which often confuses mathematicians trying to read the physics literature is a frequent shuttling back and forth between writing things in Lorentz and Euclidean signatures. The basic equations of physics do not treat space and time totally equally, in the sense that the natural metric on spacetime is a Lorentz metric, not a Riemannian one. However, in the Lorentz metric, most of the integrals one needs (such as the one computing the action) do not converge well, since one doesn't have positivity for the Lagrangian. Physicists are therefore fond of what's called Wick rotation, replacing t by it and thus "analytically continuing" from Lorentz to Euclidean signature. This results in formulations which are better behaved mathematically but not as realistic physically. Still, one can often use this to some advantage, and we will sometimes do this without further ado.

Quantum Mechanics

Unlike classical mechanics, quantum mechanics is not deterministic, only probabilistic. The key property of quantum mechanics is the Heisenberg uncertainty principle, that observable quantities are represented by *noncommuting operators* A represented on a Hilbert space $\mathcal{H}$. In the quantum world, every particle has a wave-like aspect to it, and is represented by a wave function ψ, a unit vector in $\mathcal{H}$. The phase of ψ is not directly observable, only its amplitude, or more precisely, the state

$$\varphi_\psi(A) = \langle A\psi, \psi\rangle \,.$$

But the phase is still important since *interference* depends on it.

Quantum Fields

The quantization of classical field theories is based on *path integrals*. The idea (not 100% rigorous in this formulation) is that *all fields contribute*, not just those that are critical points of the action (i.e., solutions of the classical field equations). Instead, one looks at the partition function

$$Z = \int e^{iS(\varphi)}\, d\varphi \text{ or } \int e^{-S(\varphi)}\, d\varphi \,,$$

depending on whether one is working in Lorentz or Euclidean signature. (Here we've taken $\hbar = 1$.) By the principle of stationary phase, only fields close to the classical solutions should contribute. Expectation values of physical quantities are given by

$$\langle A \rangle = \left(\int A(\varphi)\, e^{iS(\varphi)}\, d\varphi \right) / Z \,.$$

Basic Ideas of String Theory

The basic idea of string theory is to replace point particles (in conventional physics) by one-dimensional "strings." At ordinary (low) energies these strings are extremely short, on the order of the Planck length,

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.616 \times 10^{-35}\,\mathrm{m}\,.$$

A string moving in time traces out a two-dimensional surface called a worldsheet. The most basic fields in string theory are thus maps $\varphi\colon \Sigma \to X$, where Σ is a 2-manifold (the worldsheet) and X is spacetime.

String theory offers [some] hope for combining gravity with the other forces of physics and quantum mechanics.

Strings and Sigma-Models

Let Σ be a string worldsheet and X the spacetime manifold. String theory is based on the nonlinear sigma-model, where $\varphi\colon \Sigma \to X$ and the leading term in the action is

$$S(\varphi) = \frac{1}{4\pi\alpha'} \int_\Sigma \|\nabla\varphi\|^2 \, d\mathrm{vol}\,, \tag{1}$$

the energy of the map φ (in Euclidean signature). The constant α' represents (typical string length)^2 and $1/(2\pi\alpha')$ is the string tension. We have to add to this various gauge fields (giving rise to the fundamental particles) and a "gravity term" involving the scalar curvature of the metric on X. Usually we also require supersymmetry; this means the theory involves both bosons and fermions and there are symmetries interchanging the two. (But this is a subject for a different course, such as given by Freed.)

The B-Field and H-Flux

For various reasons, it's important to add to the action (1) another (Wess-Zumino) term of the form

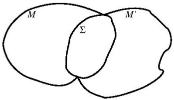
$$\frac{1}{4\pi\alpha'} \int_{\Sigma} \varphi^{*} B \,, \tag{2}$$

where B is a (locally defined) 2-form on spacetime, X. B is usually called the B-field. It need not be closed or even globally defined, just as long as it makes sense locally. (Recall the strings are really "small" in most cases.) But $H = dB$, a 3-form, should always be a well-defined closed 3-form on X, usually called the H-flux.

An interesting model for study, the *Wess-Zumino-Witten model* (WZW) has X a compact Lie group, say $SU(2)$, and H the canonical 3-form. This H is not exact so B cannot be globally defined.

Integrality of the H-Flux

For reasons that we'll discuss later, it's important to know that the de Rham class of the H-flux has to be an *integral* class. The reason for this is that if $\varphi(\Sigma)$ bounds two different oriented 3-manifolds (with boundary) M and M' in X, then



$$\frac{1}{4\pi\alpha'}\int_{M\cup -M'} H = \frac{1}{4\pi\alpha'}\int_M H - \frac{1}{4\pi\alpha'}\int_{M'} H$$

cannot contribute to the partition function or we'd have an anomaly. But the partition function involves

$$\int e^{iS(\varphi)}\, d\varphi$$

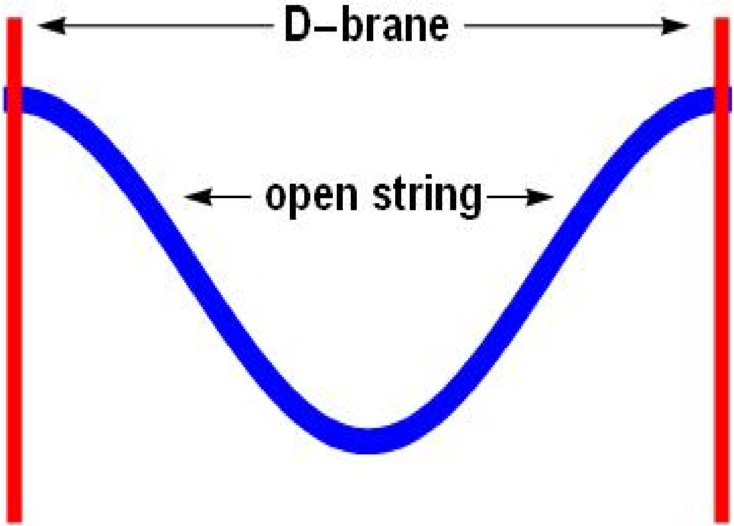
so things are OK provided $\langle H, [M \cup -M'] \rangle \in 8\pi^2\alpha'\mathbb{Z}$.

D-Branes

Physicists talk about both closed and open strings. The terminology doesn't quite match that of mathematicians. Both kinds of strings are given by compact manifolds, but in the "open" case there is a boundary. So to get a reasonable theory one has to impose boundary conditions. Usually, these are Dirichlet or Neumann conditions on some submanifold of X where the boundary of Σ must map. These submanifolds are traditionally called D-branes, "D" for *Dirichlet* and *brane* a back-formation from *membrane*. Sometimes the name D-brane is retained even without Dirichlet boundary conditions.

In post-1995 string theory, the D-branes play a fundamental role, and can even sometimes be viewed themselves as (topological) fields.

An Open String Moving on a D-Brane



The Different String Theories

There are really five different (supersymmetric) string theories, having slightly different fields and Lagrangians. We will mostly focus on types IIA and IIB. The five theories are:

- Type I. This is the one theory that involves *unoriented* strings.
- Type IIA. A theory with oriented strings where left-moving and right-moving spinors have opposite handedness.
- Type IIB. A theory with oriented strings where left-moving and right-moving spinors have the same handedness.
- E_8 Heterotic. A theory where left-movers behave as in bosonic theory and right-movers behave as in supersymmetric theory, and the gauge group is the product of two copies of the exceptional Lie group E_8.
- $SO(32)$ Heterotic. A theory where left-movers behave as in bosonic theory and right-movers behave as in supersymmetric theory, and the gauge group is [locally] the Lie group $SO(32)$.

Textbook References on String Theory

From the physics point of view:

- J. Polchinski, *String Theory*, 2 vols., Cambridge, 1998.
- B. Zwiebach, *A First Course in String Theory*, Cambridge, 2004.
- K. and M. Becker and J. Schwarz, *String Theory and M-Theory*, Cambridge, 2007.
- E. Kiritsis, *String Theory in a Nutshell*, Princeton, 2007.

From a more mathematical point of view:

- *Quantum Fields and Strings: A Course for Mathematicians*, 2 vols., Amer. Math. Soc. and IAS, 1999.
- *Mirror Symmetry*, Amer. Math. Soc. and Clay Math. Inst., 2003.

What is a Duality?

A duality is a transformation between different-looking physical theories that, rather magically, have the same observable physics. Often, such dualities are part of a discrete group, such as $\mathbb{Z}/2$ or $\mathbb{Z}/4$ or $SL(2,\mathbb{Z})$.

Example (Electric-magnetic duality)

There is a symmetry of Maxwell's equations in free space

$$\begin{aligned} \nabla\cdot E &= 0, & \nabla\cdot B &= 0, \\ \frac{\partial E}{\partial t} &= c\,\nabla\times B, & \frac{\partial B}{\partial t} &= -c\,\nabla\times E, \end{aligned} \tag{3}$$

given by $E\mapsto -B$, $B\mapsto E$. This is a duality of order 4.

Fourier Duality

Example (Configuration space-momentum space duality)

Another example from standard quantum mechanics concerns the quantum harmonic oscillator (say in one dimension). For an object with mass m and a restoring force with "spring constant" k, the Hamiltonian is

$$H = \frac{k}{2}x^2 + \frac{1}{2m}p^2 , \tag{4}$$

where p is the momentum. In classical mechanics, $p = m\dot{x}$. But in quantum mechanics (with $\hbar$ set to 1),

$$[x, p] = i . \tag{5}$$

We obtain a duality of (4) and (5) via $m \mapsto \frac{1}{k}$, $k \mapsto \frac{1}{m}$, $x \mapsto p$, $p \mapsto -x$. This is again a duality of order 4, and is closely related to the Fourier transform.

T-Duality

One of the important dualities in string theory, called T-duality ("T" for "torus"), will be the main subject of this lecture series. This duality sets up an equivalence of string theories on two very different spacetime manifolds X and $X^\sharp$. The basic idea is that tori in X are replaced by their dual tori in $X^\sharp$. In the simplest case, that means that X has a circle factor of radius R and $X^\sharp$ has a circle factor of radius $\widetilde{R} = \frac{\alpha'}{R}$. The duality also involves changes in the metric and the B-field, known as the Buscher rules, after Buscher, who derived them in 1987–88.

Derivation of T-Duality, Following Buscher

Consider the simplest case. Take Σ a closed Riemannian 2-manifold and consider the action (1) for a map to a circle, gotten by integrating a 1-form ω on Σ:

$$S(\omega) = \frac{1}{4\pi\alpha'} \int_\Sigma \frac{R^2}{\alpha'} \omega \wedge *\omega.$$

Add a new parameter θ, and consider instead

$$S(\omega, \theta) = \frac{1}{4\pi\alpha'} \int_\Sigma \left(\frac{R^2}{\alpha'} \omega \wedge *\omega + 2\theta\, d\omega \right).$$

For an extremum of S with respect to variations in θ, we need $d\omega = 0$, so we get back the original theory. But instead we can take the variation in ω.

Derivation of T-Duality (cont'd)

$$\delta S = \frac{R^2}{4\pi\alpha'^2}\int_\Sigma \left(\delta\omega \wedge *\omega + \omega \wedge *\delta\omega + \frac{2\alpha'}{R^2}\theta\, d\delta\omega\right)$$
$$= \frac{R^2}{4\pi\alpha'^2}\int_\Sigma \delta\omega \wedge \left(2 * \omega + \frac{2\alpha'}{R^2} d\theta\right),$$

so if $\delta S = 0$, $*\omega = \frac{-\alpha'}{R^2} d\theta$ and $\omega = \frac{\alpha'}{R^2} * d\theta$. If $\eta = d\theta$, substituting back into $S(\omega, \theta)$ gives

$$S'(\eta) = \frac{1}{4\pi\alpha'}\int_\Sigma \left(\frac{R^2}{\alpha'}\left(\frac{\alpha'}{R^2}\right)^2 \eta \wedge *\eta + 2\frac{\alpha'}{R^2}\theta\, d * \eta\right)$$
$$= -\frac{1}{4\pi\alpha'}\int_\Sigma \frac{\alpha'}{R^2}\eta \wedge *\eta$$

which is just like the original action (with η replacing ω, $\widetilde{R} = \frac{\alpha'}{R}$ replacing R).

Connection with Theta Functions

T-duality is also related to the classical theory of θ-functions. Consider a simple theory where $\Sigma = S^1$ and $X = \mathbb{R}/(2\pi R\mathbb{Z})$. (If you like, these are the space-like directions and there is another [inert] time direction, a factor of $\mathbb{R}$.) A string winding around X is like a wound-up rubber band; the higher the winding number, the greater the energy. For simplicity, let's just sum over the semi-classical states, the harmonic maps $x \mapsto 2\pi nRx \colon \mathbb{R}/\mathbb{Z} \to X$, instead of taking the path integral, which involves infinite-dimensional integration over *all* paths. The partition function is then:

$$Z_R = \sum_{n=-\infty}^{\infty} e^{-\pi\, n^2\, R^2/\alpha'}\,, \tag{6}$$

a classical θ-function.

Connection with Poisson Summation

Applying the identity

$$\int_{-\infty}^{\infty} e^{-2\pi\, i\, x\, y}\, e^{-s\,\pi\, x^2} dx = \frac{1}{\sqrt{s}} e^{-\,\pi\, y^2/s}$$

together with the Poisson summation formula to (6) gives the famous identity (used in the proof of the functional equation of the Riemann ζ-function):

$$Z_R = \frac{\sqrt{\alpha'}}{R}\, Z_{\widetilde{R}}\,, \tag{7}$$

where $\widetilde{R} = \alpha'/R$, which is basically T-duality.

S-Duality

Another important duality in string theory is S-duality ("S" for "strong/weak"). This duality comes from the Montonen-Olive conjecture, that says that the classical electric-magnetic duality should extend in many cases to a duality of quantum theories interchanging "electric" and "magnetic" charges. S-duality interchanges the strong coupling limit of one string theory with the weak coupling limit of another one.

It has been pointed out [Harvey, Moore, and Strominger, Reducing S-duality to T-duality, *Phys. Rev. D* (3) **52** (1995), no. 12, 7161–7167] that S-duality and T-duality are closely linked.

S-duality is related to Langlands duality between two Lie groups, and comes from the T-duality between the tori defined by the weight and coweight lattices for a Cartan subalgebra.

AdS/CFT Duality

Another duality which has attracted a lot of attention recently is often called AdS/CFT duality. (Here AdS stands for "anti de Sitter space," a spacetime manifold of constant curvature, and CFT stands for "conformal field theory.") This duality was discovered by Juan Maldacena in 1997, and in general posits an equivalence between gauge theories in dimension d (usually 4) and string theories in a spacetime of dimension $d + 1$. There is by now a huge literature on this.

M-Theory and F-Theory

There are many other dualities connected with string theory, which fit into various patterns which have been schematized like this by Schwarz, like this by Witten, like this by the Cambridge relativity group, and like this at still another web site.

Superstring theories are (to eliminate certain anomalies) required to be 10-dimensional. The dualities between them seem to involve an 11-dimensional theory, called M-theory, which reduces to *supergravity* in the low energy limit, and a 12-dimensional theory, called F-theory.

Part II

K-Theory and its Relevance to Physics

Vector Bundles

Let X be a locally compact Hausdorff space. A family of vector spaces over X is given by a continuous open surjective map $\pi\colon E \to X$, with E locally compact Hausdorff, with scalar multiplication and vector addition maps, $\mathbb{C} \times E \to E$ and $E \times_X E \to E$, satisfying certain obvious axioms. Such "families of vector spaces over X" form a category, with the morphisms given by commuting diagrams

$$\begin{array}{ccc} E_1 & \xrightarrow{\ \varphi\ } & E_2 \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & X & \end{array}$$

with φ linear on fibers. A vector bundle over X is then a family of vector spaces over X, E, which is *locally trivial*, in that there is an open covering $\{U_j\}$ of X with $E|_{U_j} \cong U_j \times \mathbb{C}^n$ in the category of families of vector spaces over U_j for each i.

Classification of Vector Bundles by Cohomology

There are two [equivalent] ways to classify vector bundles (in the category of families of vector spaces over X): using (Čech) cohomology and using homotopy theory. For simplicity let's take X compact.

The cohomology classification uses *transition functions*. If $E \xrightarrow{\pi} X$ is a vector bundle trivialized by the covering $\{U_j\}$, then on $U_j \cap U_k$ we have *two* trivializations. These need not coincide, but must be related by continuously varying automorphisms of $\mathbb{C}^n$, i.e., by a map $g_{jk}\colon U_j \cap U_k \to GL(n, \mathbb{C})$. The maps g_{jk} satisfy the cocycle identities

$$\begin{cases} g_{jk} g_{kj} = 1, \\ g_{jk} g_{k\ell} g_{\ell j} = 1, \end{cases}$$

and thus define a class in $H^1(X, \underline{GL(n, \mathbb{C})})$, where $\underline{GL(n, \mathbb{C})}$ is the sheaf of germs of $GL(n, \mathbb{C})$-valued continuous functions. For $n > 1$, this is *nonabelian cohomology*.

Classification of Line Bundles

For $n = 1$ (the case of line bundles), $GL(1, \mathbb{C}) = \mathbb{C}^\times$ and we have an exact sequence of sheaves

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \underline{\mathbb{C}} \xrightarrow{\exp} \underline{\mathbb{C}^\times} \to 1.$$

Since the sheaf $\underline{\mathbb{C}}$ is "fine" and thus has no higher cohomology, we get an exact sequence

$$0 = H^1(X, \underline{\mathbb{C}}) \to H^1(X, \underline{\mathbb{C}^\times}) \to H^2(X, \mathbb{Z}) \to H^2(X, \underline{\mathbb{C}}) = 0,$$

and thus line bundles are classified by $H^2(X, \mathbb{Z})$. In fact, one can check that one gets an isomorphism of groups

$$\operatorname{Pic} X \cong H^2(X, \mathbb{Z}), \tag{8}$$

where Pic denotes the group of line bundles, with group operation given by fiberwise tensor product over $\mathbb{C}$.

Classification of Vector Bundles by Homotopy Theory

The classification by homotopy theory is based on the fact that every vector bundle E (over a compact base X) is a direct summand in some trivial bundle $X \times \mathbb{C}^N$. Here N can be much larger than the rank n of the bundle. But this means that E can be viewed as continuous way of selecting a rank-n subspace from $\mathbb{C}^N$, or as a map $\varphi\colon X \to \mathrm{Gr}(n, N)$, where $\mathrm{Gr}(n, N)$ is the Grassmannian of n-dimensional subspaces in $\mathbb{C}^N$. Furthermore, homotopic maps $X \to \mathrm{Gr}(n, N)$ define isomorphic bundles, and isomorphic bundles give rise to homotopic maps, at least if one takes N sufficiently large. Thus we get a bijection

$$\mathrm{Vect}_n(X) \cong [X, \lim_{N\to\infty} \mathrm{Gr}(n, N)],$$

where $\mathrm{Vect}_n(X)$ is the set of isomorphism classes of rank-n vector bundles over X, and $[X, Y]$ is the set of path components of $\mathrm{Map}(X, Y)$.

The Splitting Principle

Attached to vector bundles are certain canonical cohomology classes, the Chern classes. There are several ways to define these. One method uses the classification of line bundles via H^2, along with:

Lemma (Splitting Principle)

Given a compact space X and a rank-n vector bundle E over X, there is a compact space Y and a map $f\colon\ Y \to X$, such that f^* is injective on cohomology and such that $f^*(E)$ splits as a direct sum of line bundles: $f^*(E) \cong L_1 \oplus \cdots \oplus L_n$.

This means that for many purposes, we can always pretend that a vector bundle splits into line bundles. Such a splitting for physicists corresponds to a symmetry breaking from $U(n)$ to $U(1)^n$.

Chern Classes

Definition

Let X, E, Y, and f be as in the Splitting Principle. Define $c(f^*E)$ to be $\prod_{j=1}^{n}(1 + c_1(L_j))$, where c_1 is the class in H^2 attached to a line bundle via (8). This has the form $1 + c_1(f^*E) + \cdots$, where $c_j(f^*E) \in H^{2j}(X, \mathbb{Z})$ is the j-th elementary symmetric function of the $c_1(L_j)$. Then define $c(E)$ so that $f^*c(E) = c(f^*E)$. (Of course one has to check that this makes sense.)

An alternate approach is to use the homotopy classification of rank-n vector bundles via maps to

$$BU(n) = \lim_{N\to\infty} \mathrm{Gr}(n, N).$$

One shows $H^*(BU(n), \mathbb{Z})$ is a polynomial ring on classes $c_j \in H^{2j}(BU(n), \mathbb{Z})$, and then if E over X is classified by $f\colon X \to BU(n)$, we define $c_j(E) = f^*(c_j)$.

Connections

There is another approach to Chern classes, more familiar to physicists. Let $\pi\colon\ E \to X$ be a vector bundle. Put a hermitian metric on E (a smoothly varying family of inner products on the fibers). A connection on E is a way of relating one section to another, and is defined by means of a "directional derivative" operator

$$\nabla\colon\ \Gamma^\infty(X, TX) \times \Gamma^\infty(X, E) \to \Gamma^\infty(X, E)\colon\ (Y, s) \mapsto \nabla_Y(s)$$

(here Γ^∞ stands for smooth sections, TX for the tangent bundle) satisfying

$$\nabla_Y(f \cdot s) = Yf \cdot s + f \cdot \nabla_Y(s).$$

Unlike the exterior derivative d, which satisfies $d^2 = 0$, one need not have $\nabla^2 = 0$, but ∇^2 does not involve differentiation; it is given by a two-form Θ, called the curvature of the connection.

Chern-Weil Theory

More precisely,

$$\Theta(Y, W) = \nabla_Y \nabla_W - \nabla_W \nabla_Y - \nabla_{[Y,W]},$$

and Θ is a 2-form with values in $\mathrm{End}(E)$.

Theorem (Chern-Weil)

The de Rham classes of the coefficients of the characteristic polynomial of

$$\frac{-1}{2\pi i}\Theta$$

are independent of the choice of connection and lie in the image of $H^{\mathrm{even}}(X, \mathbb{Z}) \to H^{\mathrm{even}}(X, \mathbb{R})$.

One can then check that these classes are the images of the $c_n(E)$ in $H^{\mathrm{even}}(X, \mathbb{R})$, up to sign.

The Chern Character

The total Chern class behaves well under direct sum of vector bundles:

$$c(E \oplus E') = c(E)c(E'),$$

but not so well under tensor products. So we introduce another function of the Chern classes, the Chern character, that satisfies

$$\mathrm{Ch}(E \oplus E') = \mathrm{Ch}(E) + \mathrm{Ch}(E'), \quad \mathrm{Ch}(E \otimes E') = \mathrm{Ch}(E)\,\mathrm{Ch}(E').$$

For a line bundle L, let

$$\mathrm{Ch}(L) = \exp(c_1(L)) = 1 + c_1(L) + \tfrac{1}{2}c_1(L)^2 + \cdots .$$

For general vector bundles, we use the Splitting Principle and define

$$\mathrm{Ch}(L_1 \oplus \cdots \oplus L_n) = \sum_{j=1}^{n} \mathrm{Ch}(L_j).$$

Note that the Chern character lives in $H^*(X, \mathbb{Q})$, not $H^*(X, \mathbb{Z})$.

What is K-Theory?

For many purposes (and we will see a case in physics), one wants to be able to add and subtract vector bundles. This is done using (topological) K-theory. For X a compact Hausdorff space, we define $K(X)$ to the group completion of the monoid of isomorphism classes of vector bundles over X. In other words, $K(X)$ is the set of *formal differences* $[E]-[F]$, where E and F are vector bundles over X, and where

$$[E]-[F]=[E']-[F'] \Leftrightarrow E\oplus F'\oplus G \cong E'\oplus F\oplus G \text{ for some } G. \quad (9)$$

This is an abelian group, and it becomes a commutative ring if we let

$$([E]-[F])\cdot([E']-[F'])=[E\otimes E']+[F\otimes F']-[E\otimes F']-[F\otimes E'].$$

The Chern character gives a ring homomorphism $K(X)\rightarrow H^{\text{even}}(X,\mathbb{Q})$.

Bott Periodicity

We define K-theory with compact supports for locally compact spaces by letting

$$K(X) =_{\text{def}} \ker \iota^* \colon K(X^+) \to K(\text{pt}),$$

where $X^+ = X \cup \{\infty\}$ is the one-point compactification of X, and $\iota\colon \text{pt} \to X^+$ is the inclusion of the point at ∞. With the understanding that, if X is already compact, $X^+ = X \amalg \{\infty\}$, this extends the old definition. We let $K^{-j}(X) = K(X \times \mathbb{R}^j)$.

Theorem (Bott Periodicity)

For any locally compact space X, there is a natural isomorphism $K(X) \to K(X \times \mathbb{R}^2)$.

This has the consequence that we can think of $K^*(X)$ as being $\mathbb{Z}/2$-graded.

K-Theory and Cohomology

It turns out that the functor $X \rightsquigarrow K^*(X)$ (extended to compact pairs by letting $K^*(X, A) = \ker c_*\colon\ K^*(X/A) \to K^*(\text{pt})$) becomes a cohomology theory on the category of compact (Hausdorff) spaces, or on the category of locally compact spaces and proper maps. In other words, it satisfies all the Eilenberg-Steenrod axioms except for the dimension axiom. The theory is $\mathbb{Z}/2$-graded, so there are only two groups, $K = K^0$ and $K^1 = K^{-1}$.
What's the connection with ordinary cohomology? It turns out that the Chern character $K(X) \to H^{\text{even}}(X, \mathbb{Q})$ becomes a rational isomorphism of cohomology theories, sending products in $K(X)$ to cup products in cohomology. But the torsion in $K^*(X)$ and $H^*(X, \mathbb{Z})$ can differ. For example, all torsion in $H^*(\mathbb{RP}^n, \mathbb{Z})$ has order 2, whereas the torsion in $K^*(\mathbb{RP}^n)$ has order going to infinity as $n \to \infty$.

The Atiyah-Hirzebruch Spectral Sequence

The connection between $K^*(X)$ and $H^*(X, \mathbb{Z})$ is a little more subtle, and is governed by something called the Atiyah-Hirzebruch spectral sequence.

Theorem (Atiyah-Hirzebruch)

There is a spectral sequence converging to $K^*(X)$ *with* $E_2^{p,q} = H^p(X, K^q(\text{pt}))$. *Note that* $K^q(\text{pt}) = \mathbb{Z}$ *for* q *even,* 0 *for* q *odd. The first non-zero differential is* $d_3\colon H^p(X, \mathbb{Z}) \to H^{p+3}(X, \mathbb{Z})$, *which is equal to the Steenrod operation* Sq^3.

For those unfamiliar with spectral sequences, this basically says that there is an iterative process for computing $K^*(X)$ from $H^*(X, \mathbb{Z})$, where at each stage of the process, one computes the cohomology of the result of the previous stage E_r with regard to a differential d_r (only affecting torsion).

An Analogy: Charges in Electromagnetism

- A big puzzle in classical electricity and magnetism is that while there are plenty of charged particles (electrons, etc.), no magnetically charged particles (magnetic monopoles) have ever been observed, even though their existence would not contradict Maxwell's equations.
- Another problem with classical E&M is that it doesn't explain why charges appear to be quantized, i.e., only occur in units that are integral multiples of the charge of the electron (or of the charges of [down-type] quarks).

Dirac (1931) proposed to solve both problems at once with a quantum theory of E&M that in modern terms we would call a $U(1)$ gauge theory.

The Dirac Monopole

In Dirac's theory, we assume spacetime is a 4-manifold M, say $\mathbb{R}^4 \setminus \mathbb{R} \cong \mathbb{R}^2 \times S^2$ (Minkowski space with the time trajectory of one particle taken out). The (magnetic) vector potential (A^1, A^2, A^3) and electric potential $A^0 = \phi$ of classical E&M are combined into a single entity A, a (unitary) connection on a complex line bundle L over M. Thus iA is *locally* a real-valued 1-form, and $F = i\mu dA$, μ a constant, is a 2-form encoding both of the fields E (via the $(0, j)$ components) and B (via the (j, k) components, $0 < j < k$). The Chern class $c_1(L) \in H^2(M, \mathbb{Z}) \cong \mathbb{Z}$ is an invariant of the topology of the situation. Of course, F should really be $i\mu$ times the curvature of A, and Chern-Weil theory says that the de Rham class $[F]$ is $2\pi\mu$ times the image of $c_1(L)$ in $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$. L is associated to a principal $U(1)$-bundle $P \to M$, and Dirac identifies a section of this bundle with the phase of a wave function of a charged particle in M.

Charge Quantization

In the above setup, if we integrate F over the S^2 that links the worldline we removed, we get $2\pi\mu c_1(L)$, and this is the flux of the magnetic field through S^2. So the deleted worldline can be identified with that of a magnetic monopole of charge $g = \mu c_1(L)$ in suitable units. Suppose we consider the motion of a test charge of electric charge q around a closed loop γ in M. In quantum E&M, by the Aharonov-Bohm effect, the exterior derivative is replaced by the covariant derivative (involving the vector potential A). So the phase change in the wave function is basically the holonomy of $(P \to M, A)$ around γ, or (taking $\hbar = 1$) $\exp\left(q\mu \oint_\gamma A\right)$. Since M is simply connected, γ bounds a disk D and this is $\exp\left(-iq \int_D F\right)$. Taking D in turn to be the two hemispheres in S^2, we get two answers which differ by a factor of

$$\exp\left(i\, q \int_{S^2} F\right) = e^{2\pi i\, q\, \mu\, c_1(L)}.$$

Since this must be 1, we get Dirac's quantization condition $qg \in \mathbb{Z}$.

Chan-Paton Bundles

As we indicated before, D-branes are submanifolds of the spacetime manifold X on which "open" strings are allowed to end. In superstring theories, X is a 10-dimensional Lorentz manifold, often taken to be a product of a Riemannian manifold with $\mathbb{R}$ (representing time). One often talks about D_p-branes or p-branes, the p (with values ≤ 9) representing the dimension of the space-like part of the brane. (So caution: a p-brane is really $(p+1)$-dimensional.)

The D-branes carry Chan-Paton bundles. If such a bundle has dimension n, the brane carries a $U(n)$ gauge field, a connection on the bundle. Branes and their Chan-Paton bundles are allowed to coalesce or to split apart.

Just as there are antiparticles, there are antibranes. The bundles on such branes should be viewed as having *negative* dimension.

Charges and K-Theory

Just as in the case of electromagnetism, the D-branes carry topological charges associated to the nontriviality of the Chan-Paton bundles. In the case of electromagnetism, if X is spacetime with the worldlines of the electrons removed, X admits a line bundle whose class in $\operatorname{Pic} X$ is equivalent to knowledge of the charges. The case of (type II) string theory is analogous, but the gauge theories involved are nonabelian, i.e., involve vector bundles of higher rank. Thus physicists arrived at the idea that charges should be classified by K-theory of spacetime.

The Minasian-Moore Formula

Some of the evidence for this idea comes from the Minasian-Moore formula for D-brane charges. If we are only dealing with 9-branes (those that fill all of spacetime), then the K-theory charge is just the class $[E]$ of the Chan-Paton bundle E. For branes W which are proper submanifolds of spacetime X, the embedding of W in X is also relevant. When W and X have spin^c structures (which means one can define spinors and thus a theory of fermions), Minasian and Moore found that the K-theoretic charge should be identified with $f_!([E])$, where $f\colon W \hookrightarrow X$ and $f_!$ is the Gysin map in K-theory, a "wrong way" map defined by Atiyah-Singer.

It is actually somewhat easier to think of brane charges as living in K-homology, the homology theory *dual* to K-theory. In this theory, maps go the "right way," so we just compute the K-theory charge in $K_*(W)$ and push it forward under f_*.

Topological K-Homology

A geometric realization of K-homology was given by Baum and Douglas. If X is a compact space, any K-homology class on X may be defined by a "cycle" consisting of:

1. a compact spin^c manifold W with a map $f\colon W \to X$;
2. a [virtual] vector bundle E over W.

We add cycles to make an abelian semigroup using disjoint union. Two such cycles are homologous, i.e., define the same K-homology class, if they are related by:

1. [spin^c] bordism;
2. the relation $(W, E_1, f) + (W, E_2, f) = (W, E_1 \oplus E_2, f)$;
3. "vector bundle modification" (a way of building in Bott periodicity).

Analytical K-Homology

Another realization of K-homology (i.e., another definition of cycles that yields an isomorphic theory) is due to Kasparov, and is based on generalized elliptic operators or Fredholm modules. This is a special case of Kasparov's KK-theory for C^*-algebras, in that $K_*(X)$ (X compact) is given by $KK_*(C(X), \mathbb{C})$. An (even-dimensional) K-homology cycle on X is given by a $\mathbb{Z}/2$-graded Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ with a $*$-representation of $C(X)$, and an odd bounded self-adjoint operator T such that $T^2 - 1$ and $[T, f]$, $f \in C(X)$, are compact. (The typical example is X a compact manifold, $T = D(1 + D^2)^{-1/2}$ for some self-adjoint elliptic first-order partial differential operator, like the Dirac operator.) The equivalence relation is generated by homotopy, "block addition," and the relation that $(\mathcal{H}, T)$ is trivial if T can be changed by a compact operator so that $T^2 = 1$ and $[T, f] = 0$, $f \in C(X)$.

Relating the Two Realizations of K-Homology

Now we can explain how the two realizations of K-homology are related. If W is a closed spin^c manifold, it admits a Dirac operator D. If E is a vector bundle over W and $f\colon W \to X$, we form D_E, "Dirac with coefficients in E." This defines a class in the Kasparov model of $K_*(W)$, the dimension given by the dimension of W mod 2. If $f\colon W \to X$, where X is a compact space, we send the class of (W, E, f) (in the Baum-Douglas model of K-homology) to $f_*([D_E])$ in the Kasparov model, and this gives the isomorphism.

Brane Charges in K-Homology

Now we can explain how brane charges are defined in K-homology. If W is a D-brane in X with Chan-Paton bundle E, then if f denotes the inclusion map $W \hookrightarrow X$, (W, E, f) gives a class in $K_*(X)$ via the Baum-Douglas model, provided that W is spin^c. We will see later when this condition needs to be modified, but this condition is usually needed for anomaly cancellation [Freed-Witten].

The identification of D-brane charges with K-homology classes is Poincaré dual to the identification of these charges with K-theory classes. So the two points of view are equivalent, at least under mild conditions on X.

Part III

A Few Basics of C^*-Algebras and Crossed Products

Why C^*-Algebras?

What's so important about C^*-algebras, especially for noncommutative geometry and mathematical physics?

- Compared to noncommutative algebras in general, they have a fairly rigid structure, which makes them easier to classify.
- They generalize the notion of algebras of continuous functions (on locally compact Hausdorff spaces).
- They have isometric representations on a Hilbert space, which is required by the axioms of quantum mechanics.

Banach and C^*-Algebras

An algebra A (say over $\mathbb{C}$) is called a Banach algebra if it is equipped with a complete norm (thus making A into a Banach space) with the compatibility condition

$$\|ab\| \leq \|a\| \cdot \|b\| .$$

If there is also a conjugate-linear map $a \mapsto a^*$ satisfying

$$(a^*)^* = a, \quad (ab)^* = b^* a^*, \quad \|a^*\| = \|a\| ,$$

then A is called a Banach $*$-algebra. Finally A is called a C^*-algebra if it is a Banach $*$-algebra satisfying

$$\|a^* a\| = \|a\|^2 .$$

Consequences of the Axioms

In any Banach algebra A, the spectrum of an element a is the set of $\lambda \in \mathbb{C}$ such that $a - \lambda \cdot 1$ is not invertible. (If A doesn't have a unit, first adjoin a unit to A.) This is always a compact non-empty subset of $\mathbb{C}$, and always contains 0 if A does not have a unit. The spectral radius of a is the radius of the smallest disk centered at 0 and containing the spectrum of a. This can be computed as $\lim_{n\to\infty} \|a^n\|^{1/n}$.

In a C^*-algebra A, if $a = a^*$, or even if a and a^* commute, the norm and spectral radius of a coincide. It follows from this that the norm is determined by the algebraic structure (including the $*$-operation). Thus any $*$-homomorphism between C^*-algebras is norm non-increasing, and any injective $*$-homomorphism is an isometry.

Positive Elements

Proposition

In a C^*-algebra A, if $a = a^*$, then the spectrum of a lies in $\mathbb{R}$. If A is unital and $uu^* = u^*u = 1$, then the spectrum of u lies in $\mathbb{T}$. Furthermore, the following are equivalent:

1. *$a = a^*$ and the spectrum of a lies in $[0, \infty)$.*
2. *$a = b^2$ for some $b = b^*$.*
3. *$a = y^*y$ for some $y \in A$.*

Elements $a = a^*$ are called self-adjoint. Every element is of the form $a + ib$ with a, b self-adjoint. Elements u with $uu^* = u^*u = 1$ (in a unital C^*-algebra) are called unitary. The elements A_+ with properties 1–3 above are called positive. They span A as a vector space.

Examples of C^*-Algebras

Examples

1. $A = M_n(\mathbb{C})$, $* =$conjugate transpose, $\|a\| = \max_{|\xi|=1} |a\,\xi|$.
2. $\mathcal{H}$ a Hilbert space, $A = \mathcal{L}(\mathcal{H})$ (bounded linear operators on $\mathcal{H}$),
$$\langle a\,\xi,\, \eta\rangle = \langle \xi,\, a^*\eta\rangle\,, \quad \|a\| = \sup_{\|\xi\|=1} \|a\,\xi\|\,.$$
3. $\mathcal{H}$ as above, $A = \mathcal{K}(\mathcal{H})$ (compact linear operators on $\mathcal{H}$). This algebra does not have a unit unless $\mathcal{H}$ is finite dimensional.
4. X locally compact Hausdorff, $A = C_0(X)$ (continuous functions vanishing at infinity). $\|f\| = \sup|f(x)|$, $*$=complex conjugation. This algebra has a unit exactly when X is compact.

As we will see below, these examples are *universal* in a certain sense.

Basic Theorems About C^*-Algebras

Theorem (Gelfand)

Every commutative C^*-algebra A is isometrically $*$-isomorphic to $C_0(X)$ for some locally compact Hausdorff space X. The X is unique up to homeomorphism and obtained from A as the space of maximal (modular) ideals.

Theorem (Gelfand-Naimark)

Every C^*-algebra embeds $*$-isometrically into $\mathcal{L}(\mathcal{H})$ (as a closed $*$-subalgebra) for some Hilbert space $\mathcal{H}$. One can choose $\mathcal{H}$ finite dimensional if $\dim A$ is finite.

The proofs may be found in standard books on operator algebras, such as Dixmier, Pedersen, Kadison-Ringrose.

Morita Equivalence

For many purposes, the obvious equivalence relation on C^*-algebras ($*$-isomorphism) is too strong, and one needs something a bit weaker. The correct notion is often Morita equivalence, a C^*-algebraic version (due to Rieffel) of an equivalence relation from noncommutative ring theory. Two rings A and B are Morita equivalent if their categories of left modules are equivalent, i.e., they have the "same" representation theory. Morita's Theorem says that this is the case exactly when there is an A-B bimodule ${}_AY_B$ such that the equivalence is implemented by tensoring with Y and the reverse equivalence is implemented by tensoring with the "dual." The C^*-algebraic version is similar, but one needs topological control in the form of A- and B-valued inner products on Y satisfying certain nice relations.

Characterization of the Compact Operators

As hinted above, the compact operators $\mathcal{K}(\mathcal{H})$ play a distinguished role in the theory of C^*-algebras. We often write simply $\mathcal{K}$ when $\mathcal{H}$ is separable and infinite dimensional.

Theorem

For a C^*-algebra A, the following are equivalent:

1. *A is Morita equivalent to the scalars $\mathbb{C}$.*
2. *$A \cong \mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

If A is separable, one can add:

3. *A has, up to unitary equivalence, a unique irreducible representation on a Hilbert space.*

C^* Tensor Products

Suppose A and B are C^*-algebras. Their algebraic tensor product (over $\mathbb{C}$) $A \odot B$ is clearly a $*$-algebra with $(a \otimes b)^* = a^* \otimes b^*$. Often one wants to complete $A \odot B$ to a C^*-algebra. This is not as simple a matter as one might hope, as there are usually many different C^*-algebra norms on $A \odot B$ satisfying the obvious cross-norm condition $\|a \otimes b\| = \|a\| \cdot \|b\|$. To avoid dealing with this problem, we will always use $\otimes$ to denote the spatial tensor product, which is the completion of $A \odot B$ for its obvious $*$-representation on $\mathcal{H} \bar{\otimes} \mathcal{H}'$, where $\mathcal{H}$, $\mathcal{H}'$ are Hilbert spaces on which A, resp., B act, and $\bar{\otimes}$ is the tensor product of Hilbert spaces. (It's an easy but nontrivial exercise to check that $A \otimes B$ doesn't depend on the choices of the (faithful) representations of A and B.)

Nuclearity

Aside from the spatial C^* tensor product $A \otimes B$, which comes from the *minimal* C^* cross-norm on $A \odot B$, there is also a maximal tensor product $A \otimes_{\max} B$, whose $*$-representations correspond to commuting pairs of $*$-representations of A and B. Usually these two products are different, and there may be many "intermediate" tensor products between the two. But there is a unique C^* cross-norm on $A \odot B$ if either A or B is nuclear. Commutative C^*-algebras, finite dimensional C^*-algebras, and $\mathcal{K}$ are nuclear; so are C^*-algebras generated by representations of discrete amenable groups. The class of nuclear C^*-algebras is closed under extensions, C^* tensor products, and inductive limits.

The Brown-Green-Rieffel Theorem

There are two alternate characterizations of Morita equivalence which are often useful.

Theorem (Brown-Green-Rieffel)

If A and B are C^*-algebras, then they are Morita equivalent if and only if they both embed as opposite "full corners" of another C^*-algebra C. A corner is a C^* subalgebra of the form pCp, where p is a self-adjoint idempotent in the multiplier algebra of C. It is full if CpC is dense in C. The opposite corner to pCp is $(1-p)C(1-p)$.

Theorem (Brown-Green-Rieffel)

If A and B are separable C^*-algebras, then they are Morita equivalent if and only if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$.

Projective Modules

Let A be a ring with unit. A finitely generated projective A-module P is a direct summand in A^n for some n. These modules P are especially nice; for example, $\_ \otimes_A P$ and $\mathrm{Hom}_A(P, \_)$ are exact functors (preserve exact sequences). One can make isomorphism classes of finitely generated projective modules into an abelian monoid (abelian semigroup with a 0 element) $\mathrm{Proj}\, A$ under direct sum $\oplus$. This is almost never a group since we have addition but not subtraction. But we can convert $\mathrm{Proj}\, A$ into a group $K_0(A)$ by taking its group completion or Grothendieck group.

K_0 of Rings

More precisely, $K_0(A)$ consists of objects $[P]-[Q]$, where P and Q are finitely generated projective A-modules, subject to the relations:

1. $$[P]-[Q]=[P']-[Q'] \Leftrightarrow P\oplus Q'\oplus R \cong P'\oplus Q\oplus R \text{ for some } R. \tag{10}$$
   (Compare (9) in the definition of topological K-theory!)
2. $([P]-[Q])+([P']-[Q'])=[P\oplus P']-[Q\oplus Q']$.

When A is commutative, tensor product over A makes $K_0(A)$ into a commutative ring with unit element $[A]$.

The Serre-Swan theorem

There is a close connection between topological K-theory for spaces, defined using vector bundles, and K-theory for rings. This comes from:

Theorem (Serre-Swan)

If X is a compact Hausdorff space and E is a vector bundle over X, then the space $\Gamma(E)$ of continuous sections of E is a finitely generated projective $C(X)$-module. Conversely, every finitely generated projective $C(X)$-module is the space of a sections of a vector bundle. Thus there is a natural isomorphism between $K(X)$ and $K_0(C(X))$.
More precisely, $E \mapsto \Gamma(E)$ sets up an equivalence of categories between vector bundles over X and finitely generated projective $C(X)$-modules.

K_0 as a Functor

It is easy to see that $A \mapsto K_0(A)$ is a covariant functor from unital rings to abelian groups. We extend it to *nonunital* rings via

$$K_0(A) = \ker\left(q_* \colon\ K_0(\widetilde{A}) \to K_0(\mathbb{Z}) \cong \mathbb{Z}\right).$$

Here $\widetilde{A}$, which as an abelian group is $A \oplus \mathbb{Z} \cdot 1$, is the result of adjoining a unit to A, and q is the quotient map. If A is a $\mathbb{C}$-algebra, we could just as well define $\widetilde{A}$ to be $A \oplus \mathbb{C} \cdot 1$ and we'd get the same group $K_0(A)$.

An interesting observation is that, even if we are only interested in unital rings, K_0 is functorial under nonunital ring homomorphisms. There is one important special case.

Proposition (Morita Invariance)

For any ring A, the (nonunital) inclusion $A \hookrightarrow M_n(A)$ defined by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ induces an isomorphism on K_0.

K_0 of Banach Algebras

Instead of looking at projective modules, it is sometimes easier to look at idempotents. A finitely generated projective A-module P is determined by an idempotent $e \in M_n(A)$ (projecting A^n onto P). K-theory for Banach algebras has somewhat better properties than for general rings. The reason has to do with the following:

Proposition

Suppose A is a Banach algebra and two idempotents $e, f \in M_n(A)$ are sufficiently close. Then they are homotopic through idempotents, and the associated projective A-modules $A^n e$ and $A^n f$ are isomorphic.

Corollary

On the category of Banach algebras, K_0 is a homotopy functor. That is, homotopic homomorphisms induce the same maps on K_0.

Topological K-Theory of Banach Algebras

One can now extend K_0 to a sequence of homotopy functors on the category of Banach algebras. We define

$$K_n(A) = K_0(C_0(\mathbb{R}^n, A))$$

for $n \in \mathbb{N}$ and A a Banach algebra. Note that $C_0(\mathbb{R}^n, A)$, the algebra of A-valued functions on $\mathbb{R}^n$ vanishing at infinity, is again a Banach algebra with pointwise addition and multiplication and norm $\|f\| = \sup_x \|f(x)\|$. When A is a C^*-algebra, this coincides with the C^* tensor product $C_0(\mathbb{R}^n) \otimes A$.

Theorem (Bott Periodicity)

For any Banach algebra, there is a natural isomorphism of functors $K_0 \cong K_2$ which comes from a specific Bott element in $K_0(C_0(\mathbb{R}^2))$. Thus $K_n(A)$ really only depends on n modulo 2.

Exact Sequences

Theorem

Let A be a Banach algebra with a closed ideal I and quotient A/I. Let $i\colon I \hookrightarrow A$ be the inclusion, $q\colon A \to A/I$ be the quotient map. Then there is a natural long exact sequence

$$\cdots \xrightarrow{i_*} K_1(A) \xrightarrow{q_*} K_1(A/I) \xrightarrow{\partial} K_0(I) \xrightarrow{i_*} K_0(A) \xrightarrow{q_*} K_0(A/I)\,.$$

Because of Bott Periodicity, this sequence closes up to make an "exact hexagon."

This theorem says that K_* behaves very much like a *homology theory* on Banach algebras. If we dualize to spaces by letting $A = C_0(X)$ with X locally compact, $A/I = C_0(Y)$ for Y closed in X, then this becomes the long exact cohomology sequence in topological K-theory, since $X \rightsquigarrow C_0(X)$ is contravariant.

Stability

Because of the fact that close idempotents in a Banach algebra are equivalent, one can show that if B is a C^* inductive limit of algebras B_n (this is the completion of the algebraic inductive limit in the obvious C^* norm), then $K_0(B) = \varinjlim K_0(B_n)$. Applying this with $B_n = M_n(A)$ for some other C^*-algebra A, and observing that $\varinjlim M_n(A) = A \otimes \mathcal{K}$, we deduce a *topological* form of Morita invariance:

Theorem (Topological Morita Invariance)

For any C^*-algebra A, the (nonunital) inclusion $A \hookrightarrow A \otimes \mathcal{K}$ defined by $a \mapsto a \otimes e$, e a rank-one self-adjoint projection, induces an isomorphism on all topological K-groups.

Group Actions on C^*-Algebras

Definition

Let A be a C^*-algebra. We denote by $\operatorname{Aut} A$ the group of $*$-automorphisms of A (algebra automorphisms preserving the $*$-operation). This is a topological group with the topology of pointwise convergence. (When $A = C_0(X)$, $\operatorname{Aut} A$ is the group of homeomorphisms of X with the compact-open topology.) If G is a locally compact group, an action of G on A or C^* dynamical system is a continuous homomorphism $\alpha\colon\ G \to \operatorname{Aut} A$.

Note: While one can also consider the norm topology on $\operatorname{Aut} A$, it is usually too strong to be useful. For example, the left translation action of G on $A = C_0(G)$ is usually not continuous for the norm topology on $\operatorname{Aut} A$. But this action is continuous for the topology of pointwise convergence.

Covariant Pairs of Representations

Let $\alpha\colon G \to A$ be an action of a locally compact group on a C^*-algebra A. A covariant pair of representations of (A, G) consists of the following:

1. a Hilbert space $\mathcal{H}$,
2. a strongly continuous unitary representation U of G on $\mathcal{H}$,
3. a $*$-homomorphism φ of A satisfying

$$U(g)\varphi(a)U(g)^* = \varphi(\alpha_g(a)), \quad g \in G, \quad a \in A.$$

Examples

1. $A = \mathbb{C}$, a unitary representation of G;
2. $A = C_0(G)$, G acting by left translation, U the left regular representation of G on $L^2(G)$, φ the action on $C_0(G)$ on $L^2(G)$ by pointwise multiplication.

Crossed Products

Given an action $\alpha\colon\ G \longrightarrow A$ of a locally compact group on a C^*-algebra, there is a unique C^*-algebra whose $*$-representations are in natural bijection with the covariant pairs of representations of (A, G). It is called the crossed product and denoted $A \rtimes_\alpha G$ or $C^*(G, A, \alpha)$. When G is discrete and A is unital, $A \rtimes_\alpha G$ is generated by a copy of A and unitary elements u_g, $g \in G$, such that $u_g\, a\, u_g^* = \alpha_g(a)$. It is the completion of the finite linear combinations $\sum_g u_g\, a_g$, $a_g \in A$, in the greatest C^* norm.

Examples

1. $A = \mathbb{C}$, $A \rtimes G$ is the (maximal) group C^*-algebra $C^*(G)$ of G, the largest C^*-algebra completion of the convolution algebra $L^1(G)$.
2. $A = C_0(G)$, G acting by left translation, $C_0(G) \rtimes G \cong \mathcal{K}(L^2(G))$ by the Stone-von Neumann-Mackey Theorem.

Dual Actions

Suppose G is a locally compact abelian group. Its dual group is $\widehat{G} = \mathrm{Hom}(G, \mathbb{T})$. Pontrjagin duality says that the dual of $\widehat{G}$ is G again.

Examples

1. $G = \mathbb{Z}$, $\widehat{G} = \mathbb{T}$.
2. *Vector groups* $\mathbb{R}^n$ are self-dual. However it is better to identify the Pontrjagin dual with the dual vector space.

If G as above acts on a C^*-algebra A, there is a dual action $\widehat{\alpha}$ of $\widehat{G}$ on $A \rtimes G$ which when G is discrete is given by $\widehat{\alpha}_\gamma(u_g\, a) = \langle \gamma, g \rangle u_g\, a$. (The action is isometric since $\langle \gamma, g \rangle$ has absolute value 1.)

Takai Duality

The following result generalizes the Stone-von Neumann-Mackey Theorem.

Theorem (Takai)

Suppose α is an action of a locally compact abelian group on a C^*-algebra A. Let $\widehat{\alpha}$ be the dual action on the crossed product. Then

$$(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G} \cong A \otimes \mathcal{K}(L^2(G)) .$$

Furthermore, the double dual action can be identified with $\alpha \otimes \operatorname{Ad} \lambda$, where $(\operatorname{Ad} \lambda)(g)(a) = \lambda_g a \lambda_g^*$, λ the left regular representation.

Connes' Thom Isomorphism Theorem

For computing K-theory of crossed products, the following is quite useful.

Theorem (Connes)

Let A be a C^*-algebra equipped with an action of $\mathbb{R}$. Then the crossed product $A \rtimes \mathbb{R}$ has the same K-theory as if the action were trivial, i.e., there is a functorial isomorphism $K_{i+1}(A) \xrightarrow{\cong} K_i(A \rtimes \mathbb{R})$.

Part IV

Continuous-Trace Algebras and Twisted K-Theory

The Trace Function

We want to introduce a particularly nice class of C^*-algebras, called algebras with continuous trace. Roughly speaking, these are algebras that look something like $C(X) \otimes M_n(\mathbb{C})$ (in the unital case) or $C_0(X) \otimes \mathcal{K}$ (in the nonunital case).

Definition

Let A be a C^*-algebra. The spectrum of A, denoted $\widehat{A}$, is the set of unitary equivalence classes of irreducible $*$-representations of A on Hilbert spaces. This is a topological space with the Fell topology, defined by pointwise convergence of matrix coefficients $a \mapsto \langle \pi(a)\,\xi, \eta \rangle$.

Given $a \in A_+$, its trace function $\widehat{A} \to [0, \infty]$ is defined by $[\pi] \mapsto \operatorname{Tr} \pi(a)$. (The RHS only depends on the unitary equivalence class of π.) This function is lower semi-continuous, not necessarily continuous.

Fell's Condition

Fell characterized those C^*-algebras for which there are lots of elements with finite and continuous trace function.

Theorem

Let A be a C^*-algebra. Then the following conditions are equivalent:

1. *Elements $a \in A_+$ with finite and continuous trace function are dense.*
2. *$\widehat{A}$ is Hausdorff and A has lots of local rank-one projections, in that for every $[\pi] \in \widehat{A}$, there is an element $a \in A_+$ with $\sigma(a)$ a rank-one self-adjoint projection (in the Hilbert space of σ) for every $[\sigma]$ in a neighborhood of $[\pi]$ in $\widehat{A}$.*

The Theory of Dixmier-Douady

A C^*-algebra A satisfying the equivalent conditions of the previous theorem is called a C^*-algebra with continuous trace or a continuous-trace algebra (CT-algebra). The structure of these algebras was investigated by Dixmier and Douady.

Theorem (Dixmier-Douady)

Let X be a second countable locally compact Hausdorff space, and let A be a separable continuous-trace algebra with spectrum X. Then A is isomorphic to the algebra $\Gamma_0(X, \mathcal{A})$ of sections vanishing at ∞ of a locally trivial bundle $\mathcal{A}$ with fibers isomorphic to $\mathcal{K}$, provided that either

1. *A is stable, i.e., $A \cong A \otimes \mathcal{K}$, or*
2. *X is finite dimensional and each irreducible $*$-representation of A is of dimension $\aleph_0$.*

The Dixmier-Douady Class and Bundle Theory

Whether or not A is locally trivial in the sense of the Dixmier-Douady Theorem, Dixmier-Douady showed A has a characteristic class $\delta(A) \in H^3(X, \mathbb{Z})$. This class doesn't change if we replace A by $A \otimes \mathcal{K}$. One may explain this Dixmier-Douady class as follows:

Suppose for simplicity that A is locally trivial and comes from a bundle $\mathcal{A}$. This bundle has fibers $\mathcal{K}$ and structure group $\operatorname{Aut} \mathcal{K} \cong PU(\mathcal{H}) = U(\mathcal{H})/\mathbb{T}$, where $\dim \mathcal{H} = \aleph_0$. But $U(\mathcal{H})$ is contractible for $\mathcal{H}$ infinite dimensional. So PU has the homotopy type $B\mathbb{T}$, which is a $K(\mathbb{Z}, 2)$ space. And principal PU-bundles over X are classified by

$$[X, BPU] = [X, K(\mathbb{Z}, 3)] = H^3(X, \mathbb{Z}).$$

Thus every class in H^3 comes from a stable CT-algebra.

Gerbes

There is an alternative approach to the theory of continuous-trace algebras and the Dixmier-Douady class via the theory of gerbes, as discussed in the short survey by Hitchin. The main advantage of this approach is that it meshes well with the theory of the H-flux in string theory. Recall that we pointed out before that this is always given by a class in $H^3(X, \mathbb{Z})$, where X is spacetime. This class is the Dixmier-Douady class of a gerbe, which in turn determines a CT-algebra. The gerbe is precisely what is needed to give a rigorous definition of the Wess-Zumino term in the string action, without vague references to "locally defined" differential forms.

Review: The Brauer Group of a Field

Before we get to the next topic, it's convenient to review some classical algebra. Suppose F is a field (this time in the sense of commutative algebra, not in the sense of physics!). By Wedderburn's Theorem, every finite dimensional central simple algebra over F is of the form $M_n(D)$, where D is a division algebra having F as its center. Two such algebras are F-Morita equivalent if and only if the associated division algebras D are isomorphic.

The Morita equivalence classes $[A]$ of central simple algebras A over F form an abelian group under the operation $\otimes_F$, with F as identity element and $[A^\circ]$ (A° the same underlying vector space as A, but with the order of multiplication reversed) as the inverse of $[A]$. This is called the Brauer group $\operatorname{Br} F$. It can be shown to be isomorphic to $H^2(\operatorname{Gal}(F^s/F), (F^s)^\times)$, F^s the separable closure of F.

The Brauer Group of a Commutative Ring

Similarly there is a notion of Brauer group $\mathrm{Br}\, R$ when R is a commutative ring. This time central simple algebras are replaced by Azumaya algebras or *central separable algebras*, R-algebras A with R as center for which A is finitely generated projective as a module over $A \otimes_R A^\circ$. The R-Morita equivalence classes of these algebras again form a Brauer group $\mathrm{Br}(R)$, with the group operation as tensor product over R and $[A]^{-1} = [A^\circ]$. This generalizes the Brauer group for fields.

The Grothendieck-Serre Theorem

The special case of $\operatorname{Br} R$ with $R = C(X)$ was studied by Grothendieck and Serre, who found an analogue of the Galois cohomology computation of the Brauer group for a field.

Theorem (Grothendieck-Serre)

Let X be a connected finite CW-complex. Then the Brauer group of $C(X)$ can naturally be identified with the torsion subgroup of $H^3(X, \mathbb{Z})$, and the Azumaya algebras over $C(X)$ are all of the form $\Gamma(X, \mathcal{A})$, where $\mathcal{A}$ is a locally trivial bundle of algebras over X with fibers $M_n(\mathbb{C})$ and structure group $\operatorname{Aut} M_n \cong PGL(n, \mathbb{C})$.

Green's Topological Brauer Group

When X is a connected finite CW-complex, the Azumaya algebras over $C(X)$ are precisely the unital CT-algebras with center $C(X)$. Philip Green proposed taking a broader point of view: allowing X to be locally compact and considering *all* CT-algebras A with spectrum $\widehat{A} \cong X$, viewed as algebras over $C_0(X)$. If one then considers these algebras up to $C_0(X)$-linear (topological) Morita equivalence, they again form a Brauer group $\operatorname{Br} X$ with group operation given by $\otimes_{C_0(X)}$ (topological tensor product).

Theorem (Green)

Let X be a second countable locally compact Hausdorff space. Then the Dixmier-Douady class defines an isomorphism $\operatorname{Br} X \cong H^3(X, \mathbb{Z})$. When X is a finite CW-complex, the Grothendieck-Serre Brauer group embeds as the torsion subgroup.

An Analogy: Cohomology with Local Coefficients

To explain the idea of twisted K-theory, it helps to think of an analogous (but simpler) theory: cohomology with local coefficients. (Čech) cohomology can be identified with the sheaf cohomology of a *constant* sheaf of abelian groups. If we replace this by a *locally constant* sheaf or local coefficient system, we get cohomology with local coefficients. For example, an oriented compact n-manifold M satisfies Poincaré duality $H^i(M, \mathbb{Z}) \xrightarrow{\cong} H_{n-i}(M, \mathbb{Z})$. If M isn't orientable, there is a canonical local coefficient system $\underline{\mathbb{Z}}$ that has fiber $H_n(M, M \smallsetminus \{x\})$ at $x \in M$, and we instead have Poincaré duality with local coefficients $H^i(M, \underline{\mathbb{Z}}) \xrightarrow{\cong} H_{n-i}(M, \mathbb{Z})$.

Twisted K-Theory

In a similar way, if an n-manifold has a spin^c structure, it satisfies Poincaré duality in K-theory, $K^i(M) \xrightarrow{\cong} K_{n-i}(M)$. If there is no spin^c structure, we need twisted K-theory. The simplest way to define this is, given a class $H \in H^3(M, \mathbb{Z})$, to take the K-theory of a *noncommutative* C^*-algebra, namely the stable continuous-trace algebra $CT(M, H)$ with H as its Dixmier-Douady class. Thus we define $K^{-i}(M, H) = K_i(CT(M, H))$.

Examples

1. $M = SU(3)/SO(3)$, $\pi_1(M) = 0$, $H_2(M) \cong H^3(M) \cong \mathbb{Z}/2$. One has Poincaré duality for twisted K-theory.
2. If $H = 0$, then $CT(M, 0) = C_0(M) \otimes \mathcal{K}$, and $K^{-i}(M, H) = K_i(C_0(M) \otimes \mathcal{K}) \cong K_i(C_0(M)) \cong K^{-i}(M)$ by Morita invariance. So twisted K-theory generalizes ordinary K-theory.

The Atiyah-Hirzebruch Spectral Sequence

Recall what we said before about how to compute K-theory from ordinary cohomology. Something quite similar is true in the case of twisted K-theory $K^*(X, H)$, $H \in H^3(X, \mathbb{Z})$, except that the differentials in the spectral sequence now involve H as well. The first non-zero differential is $d_3 \colon H^p(X, \mathbb{Z}) \to H^{p+3}(X, \mathbb{Z})$, which is equal to the sum of the Steenrod operation Sq^3 and cup product with H.

Example

$X = S^3$, $H = k \neq 0$ (when we identify $H^3(S^3)$ with $\mathbb{Z}$). In this case $\mathrm{Sq}^3 = 0$ but $d_3 \colon \mathbb{Z} \cong H^0(X) \to H^3(X) \cong \mathbb{Z}$ is multiplication by k. So we get $K^0(S^3, H) = 0$ and $K^1(S^3, H) \cong \mathbb{Z}/k$.

Twisted K-Theory and String Theory

- So what does any of this have to do with string theory and string duality?
- Well, what we said before about brane charges being classified by K-theory is not exactly right. This is true when the H-flux is trivial, but not in general.
- By Freed-Witten, in type II string theory, W_3 of a stable D-brane must match the restriction of the H-flux class.
- In general type II string theory, the brane changes take values in $K^*(X, H)$, the Ramond-Ramond charges in the even group in type IIB and in the odd group in type IIA.

Part V

The Topology of T-Duality and the Bunke-Schick Construction

Topology Change in T-Duality

We talked about the simplest case of T-duality in the first lecture. There, string theory on $X = Z \times T$, where $T = S^1$ is a circle of radius R, corresponds to a dual string theory on $X^\sharp = Z \times T^\sharp$, where $T^\sharp$ is the dual circle with radius $\widetilde{R} = \frac{\alpha'}{R}$. But what if X is fibered by circles, but doesn't split as a product?
The first example of this phenomenon was studied by Alvarez, Alvarez-Gaumé, Barbón, and Lozano in 1993. Their discovery was generalized 10 years later by Bouwknegt, Evslin, and Mathai. Let's start with the simplest nontrivial example of a circle fibration, where $X = S^3$, identified with $SU(2)$, T is a maximal torus. Then T acts freely on X (say by right translation) and the quotient X/T is $\mathbb{CP}^1 \cong S^2$, with quotient map $p\colon X \to S^2$ the Hopf fibration. Assume for simplicity that the B-field vanishes.

The Case of S^3

Let's examine this case in more detail. We have $X = S^3$ fibering over $Z = X/T = S^2$. Think of Z as the union of the two hemispheres $Z^{\pm} \cong D^2$ intersecting in the equator $Z^0 \cong S^1$. The fibration is trivial over each hemisphere, so we have $p^{-1}(Z^{\pm}) \cong D^2 \times S^1$, with $p^{-1}(Z^0) \cong S^1 \times S^1$. So the T-dual also looks like the union of two copies of $D^2 \times S^1$, joined along $S^1 \times S^1$. However, we have to be careful about the clutching that identifies the two copies of $S^1 \times S^1$. In the original Hopf fibration, the clutching function $S^1 \to S^1$ winds once around, with the result that the fundamental group $\mathbb{Z}$ of the fiber T dies in the total space X. But T-duality is supposed to interchange "winding" and "momentum" quantum numbers. So $X^\sharp$ has no winding and is just $S^2 \times S^1$.

So what happened to the clutching function? It shows up in the H-flux of the dual!

T-Duality with a B-field

To explain this, let's go back to Buscher's derivation of T-duality for the sigma-model with maps $x = (x_1, x_0)\colon \Sigma \to X^\sharp = Z \times S^1$, but this time including the Wess-Zumino term. The action now has the form

$$S(x) = \frac{1}{4\pi\alpha'} \int_\Sigma \left(\|\nabla x_1\|_Z^2 \, d\mathrm{vol}_\Sigma + \frac{R^2}{\alpha'} dx_0 \wedge * dx_0 + x^* B \right) . \quad (11)$$

When we dualize the S^1, we have to be careful about the part of B that involves this factor.

In our situation, we are starting with a case where $B^\pm = \eta^\pm \times d\mathrm{vol}_{S^1}$ is a 2-form over $Z^\pm \times S^1$, and $dB^\pm$ is a volume form on $Z^\pm \times S^1$. Note that $dB^\pm$, but not $B^\pm$, are supposed to agree on $Z^0 \times S^1$.

T-Duality with a B-field (cont'd)

In terms of the closed 1-form $\omega = dx_0$, the action becomes

$$S(\omega) = \frac{1}{4\pi\alpha'} \int_\Sigma \cdots + \frac{R^2}{\alpha'} \omega \wedge *\omega + \omega \wedge x_1^* \beta,$$

where we've left out terms not involving $x_0 \colon \Sigma \to S^1$, since they don't change under T-duality. As before, introduce the Lagrange multiplier θ to get

$$S(\omega, \theta) = \frac{1}{4\pi\alpha'} \int_\Sigma \cdots + \frac{R^2}{\alpha'} \omega \wedge *\omega + \omega \wedge x_1^* \beta + 2\theta \, d\omega,$$

which if we vary θ gives back the original action. But take the variation in ω instead.

T-Duality with a B-field (cont'd)

We set $\delta S = 0$ and get what we had before but with an extra term:

$$\frac{2R^2}{\alpha'} * \omega + x_1^* \beta + 2d\theta = 0.$$

So $*\omega = \frac{-\alpha'}{R^2}\left(d\theta + \frac{1}{2}x_1^*\beta\right)$ and $\omega = \frac{\alpha'}{R^2} * \left(d\theta + \frac{1}{2}x_1^*\beta\right)$. If $\eta = d\theta$, substituting back into $S(\omega, \theta)$ gives

$$S'(\eta) = \frac{-1}{4\pi\alpha'} \int_\Sigma \cdots + \frac{\alpha'}{R^2}\left(\eta + \frac{1}{2}x_1^*\beta\right) \wedge * \left(\eta + \frac{1}{2}x_1^*\beta\right),$$

which has the same form as S except that $R \leftrightarrow \frac{\alpha'}{R}$ and η is shifted by $\frac{1}{2}x_1^*\beta$. Recall β is not globally defined; the forms $\beta^\pm$ differ by a closed 1-form on Z^0.

K-Theory Matching

Thus when we apply T-duality starting with $X^\sharp = Z \times S^1$ and the H-flux a generator of $H^3(X^\sharp)$, we see the closed 1-form associated to the T-dual is shifted on one hemisphere relative to the another, the shifting associated to a generator of $H^1(Z^0)$. That shows exactly that the clutching map of the dual theory on X corresponds to the identity map $S^1 \to S^1$, and so the dual X is not $S^2 \times S^1$ but S^3.

We can also explain this in terms of matching of D-brane charges. If the sigma models on X and $X^\sharp$ are to give indistinguishable physics, the D-brane charges in the two theories must live in isomorphic groups.

Thus we want to require $K^*(X, H) \cong K^{*+1}(X^\sharp, H^\sharp)$. The degree shift comes from interchange of type IIA string theory with type IIB.

The Case of $S^2 \times S^1$ and S^3

Let's check this principle of K-theory matching in the case we've been considering, $X = S^3$ fibered by the Hopf fibration over $Z = S^2$. The H-flux on X is trivial, so D-brane changes lie in $K^*(S^3)$, with no twisting. And $K^0(S^3) \cong K^1(S^3) \cong \mathbb{Z}$.
On the T-dual side, we expect to find $X^\sharp = S^2 \times S^1$, also fibered over S^2, but simply by projection onto the first factor. If the H-flux on X were trivial, D-brane changes would lie in $K^0(S^2 \times S^1)$ and $K^1(S^2 \times S^1)$, both of which are isomorphic to $\mathbb{Z}^2$, which is too big.
On the other hand, we can compute $K^*(S^2 \times S^1, H^\sharp)$ for the class $H^\sharp$ which is k times a generator of $H^3 \cong \mathbb{Z}$, using the Atiyah-Hirzebruch Spectral Sequence. The differential is

$$H^0(S^2 \times S^1) \xrightarrow{k} H^3(S^2 \times S^1),$$

so when $k = 1$, $K^*(S^2 \times S^1, H^\sharp) \cong K^*(S^3) \cong \mathbb{Z}$ for both $* = 0$ and $* = 1$.

Axioms for Topological T-Duality

- This discussion suggests we should try to develop an axiomatic treatment of the topological aspects of T-duality. Note that we are ignoring many things, such as the underlying metric on spacetime and the auxiliary fields.
- Axioms:
  - We have a suitable class of spacetimes X each equipped with a principal S^1-bundle $X \to Z$. (X might be required to be a smooth connected manifold.)
  - For each X, we assume we are free to choose any H-flux $H \in H^3(X, \mathbb{Z})$.
  - There is an involution (map of period 2) $(X, H) \mapsto (X^\sharp, H^\sharp)$ keeping the base Z fixed.
  - $K^*(X, H) \cong K^{*+1}(X^\sharp, H^\sharp)$.

The Bunke-Schick Construction

Bunke and Schick suggested constructing a theory satisfying these axioms by means of a universal example. It is known that (for reasonable spaces X, say CW complexes) all principal S^1-bundles $X \to Z$ come by pull-back from a diagram

$$\begin{array}{ccc} X & \dashrightarrow & ES^1 \simeq * \\ \downarrow & & \downarrow \\ Z & \dashrightarrow & BS^1 \simeq K(\mathbb{Z}, 2) \end{array}$$

Here the map $Z \dashrightarrow K(\mathbb{Z}, 2)$ is unique up to homotopy, and pulls the canonical class in $H^2(K(\mathbb{Z}, 2), \mathbb{Z})$ back to c_1 of the bundle.

Similarly, every class $H \in H^3(X, \mathbb{Z})$ comes by pull-back from a canonical class via a map $X \dashrightarrow K(\mathbb{Z}, 3)$ unique up to homotopy.

The Gysin Sequence

For future use, let's review a classical theorem from algebraic topology.

Theorem (Gysin sequence)

Let $X \xrightarrow{p} Z$ be a principal S^1-bundle over a path-connected base Z with Chern class $c \in H^2(Z, \mathbb{Z})$. Then the cohomology groups of X and Z are related by a long exact Gysin sequence

$$\cdots \to H^k(Z,\mathbb{Z}) \xrightarrow{\cup c} H^{k+2}(Z,\mathbb{Z}) \xrightarrow{p^*} H^{k+2}(X,\mathbb{Z}) \xrightarrow{p_!} H^{k+1}(Z,\mathbb{Z}) \xrightarrow{\cup c} H^{k+3}(Z,\mathbb{Z}) \to \cdots . \quad (12)$$

The Bunke-Schick Theorem

Theorem (Bunke-Schick)

There is a classifying space R, unique up to homotopy equivalence, with a fibration

$$\begin{array}{ccc} K(\mathbb{Z},3) & \longrightarrow & R \\ & & \downarrow \\ & & K(\mathbb{Z},2)\times K(\mathbb{Z},2), \end{array} \tag{13}$$

and any $(X,H)\to Z$ as in the axioms comes by a pull-back

$$\begin{array}{ccc} X & \dashrightarrow & E \\ \downarrow & & \downarrow p \\ Z & \dashrightarrow & R, \end{array}$$

with the horizontal maps unique up to homotopy and H pulled back from a canonical class $h\in H^3(E,\mathbb{Z})$.

The Bunke-Schick Theorem (cont'd)

Theorem (Bunke-Schick)

Furthermore, the k-invariant of the Postnikov tower (13) characterizing R is the cup-product in

$$H^4(K(\mathbb{Z},2)\times K(\mathbb{Z},2),\mathbb{Z})$$

of the two canonical classes in H^2. The space E in the fibration

$$\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \Big\downarrow p \\ & & R \end{array}$$

has the homotopy type of $K(\mathbb{Z},3)\times K(\mathbb{Z},2)$.

Sketch of Proof of the Bunke-Schick Theorem

Let E be the free loop space

$$E = \Lambda K(\mathbb{Z}, 3) = \mathrm{Map}(S^1, K(\mathbb{Z}, 3)),$$

on which S^1 acts by rotating the domain S^1. The "Borel construction" gives a fibration

$$\begin{array}{ccc} E & \xrightarrow{p} & R = ES^1 \times_{S^1} E \\ & & \Big\downarrow c \\ & & BS^1 = K(\mathbb{Z}, 2). \end{array} \tag{14}$$

We can think of c as the Chern class of a circle bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \Big\downarrow p \\ & & R. \end{array} \tag{15}$$

Sketch of Proof of the Bunke-Schick Theorem (cont'd)

The free loop space comes with a fibration

$$\begin{array}{ccc} \Omega K(\mathbb{Z},3) = K(\mathbb{Z},2) & \longrightarrow & E = \Lambda K(\mathbb{Z},3) \\ & & \downarrow e \\ & & K(\mathbb{Z},3), \end{array}$$

where $\Omega K(\mathbb{Z},3)$ is the *based* loop space and e is evaluation of loops at $1 \in S^1$. Since e has a section, given by constant loops, $E \simeq K(\mathbb{Z},2) \times K(\mathbb{Z},3)$ and we have a canonical class $h \in H^3(E,\mathbb{Z})$ (associated to the map e).

We just need to check that E and R have the right properties.

Sketch of Proof of the Bunke-Schick Theorem (cont'd)

Given a pair (X, H) over a base Z, we have a map $Z \dashrightarrow BS^1$ classifying $X \to Z$. This comes from an S^1-equivariant map $X \overset{S^1}{\dashrightarrow} ES^1$. We also have a map $X \dashrightarrow K(\mathbb{Z}, 3)$ classifying H. View this as an equivariant map $X \overset{S^1}{\dashrightarrow} \Lambda K(\mathbb{Z}, 3)$ and take the product to get a commuting diagram

$$\begin{array}{ccc} X & \dashrightarrow & ES^1 \times E \simeq E \\ \downarrow & & \downarrow p \\ Z & \dashrightarrow & R. \end{array}$$

It's easy to see that this identifies $(X, H) \to Z$ with the pull-back of $(E, h) \xrightarrow{p} R$.

The Homotopy Type of R

From the fibration

$$\begin{array}{ccc} E & \xrightarrow{p} & R \\ & & \downarrow c \\ & & K(\mathbb{Z},2) \end{array}$$

and the fact that $E \simeq K(\mathbb{Z},2) \times K(\mathbb{Z},3)$, we see $\pi_2(R) \cong \mathbb{Z}^2$ and $\pi_3(R) \cong \mathbb{Z}$. (One can also see this by computing the set of pairs (X, H) living over S^n for each n.) So Postnikov theory gives a fibration of the form (13). This fibration is pulled back from the universal $K(\mathbb{Z},3)$ fibration

$$\begin{array}{ccc} K(\mathbb{Z},3) = \Omega K(\mathbb{Z},4) & \longrightarrow & \text{pt} \\ & & \downarrow \\ & & K(\mathbb{Z},4) \end{array}$$

by the k-invariant $K(\mathbb{Z},2)^2 \to K(\mathbb{Z},4)$.

The Homotopy Type of R (cont'd)

If the k-invariant were trivial, we'd have $R \simeq K(\mathbb{Z},2)^2 \times K(\mathbb{Z},3)$, which would be impossible, since it would imply the H-flux H on X is always the pull-back of a class in $H^3(Z,\mathbb{Z})$, which contradicts the Gysin sequence.
On the other hand, the k-invariant is certainly trivial on one of the copies of $K(\mathbb{Z},2)$, because the composite

$$K(\mathbb{Z},2) \xrightarrow{\text{inj. of one summand}} E \simeq K(\mathbb{Z},2) \times K(\mathbb{Z},3) \to R$$

splits one of the maps $R \to K(\mathbb{Z},2)$. The fact that the k-invariant has to be the product of the two canonical classes in H^2 follows from the Gysin sequence applied to the circle bundle (15).

Topological T-Duality, Revisited

Now we can see how topological T-duality arises. R has a homotopy automorphism of period 2 that comes from interchanging the two copies of $K(\mathbb{Z}, 2)$ in (13). (That's OK since the k-invariant is symmetric.)

In terms of the fibration $\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \downarrow{\scriptstyle c} \\ & & R \end{array}$, the first copy of $K(\mathbb{Z}, 2)$ is the Chern class of c, and the second copy comes from $c_!(h)$, where $c_!\colon\ H^3(E) \to H^2(R)$ is the Gysin map or "integration along the fiber." So T-duality interchanges these two. We will see this from another point of view later.

The Case of $S^2 \times S^1$ and S^3, Revisited

We conclude this lecture with the case of $S^2 \times S^1$ and S^3 again. Let $\alpha \in H^2(S^2)$, $\beta \in H^1(S^1)$ be the usual generators. Look at the diagram

$$\begin{array}{ccc} (X, H) = (S^3, 0) & & (X^\sharp, H^\sharp) = (S^2 \times S^1, \alpha \times \beta) \\ & \searrow^{p} \quad {}^{p^\sharp}\swarrow & \\ & Z & \end{array}$$

We have $c_1(p) = (p^\sharp)_!(H^\sharp) = \alpha$, $c_1(p^\sharp) = p_!(H) = 0$. So indeed T-duality interchanges

$$c_1(p) \leftrightarrow (p^\sharp)_!(H^\sharp), \quad c_1(p^\sharp) \leftrightarrow p_!(H).$$

Part VI

T-Duality via Crossed Products

What's Coming Next?

- In the last lecture, we proposed a set of axioms for topological T-duality, and showed via the Bunke-Schick Theorem that there is an essentially unique way of satisfying the first three axioms.
- The problem is to check the final axiom about preservation of twisted K-theory under duality.
- In this lecture, we will use the K-theory of crossed products to verify the last axiom.
- In this way, we will arrive at a surprising unification of the three areas of topology, C^*-algebras, and string duality.

Strategy of the Method

Suppose X is a spacetime manifold which is a principal S^1-bundle over Z. Then we have a free action of S^1 on X, and thus an action $\mathbb{T} \to \mathrm{Aut}\, C_0(X)$. In order to use Connes' Thom Isomorphism Theorem, we lift this action to the universal cover $\mathbb{R}$ of $\mathbb{T}$ and think of it as an action $\alpha\colon \mathbb{R} \to \mathrm{Aut}(C_0(X))$ which is trivial on $\mathbb{Z}$. Form the crossed product $A = C_0(X) \rtimes_\alpha \mathbb{R}$. By Connes' Theorem, we have $K_{*+1}(A) \cong K_*(C_0(X)) \cong K^*(X)$. Now suppose we knew for some reason that A is a continuous-trace algebra over some space $X^\sharp$, with Dixmier-Douady invariant $H^\sharp$, and suppose we knew that $X^\sharp$ was also a principal S^1-bundle over Z. Then we'd have

$$K^{*+1}(X^\sharp, H^\sharp) \cong K^*(X),$$

which is the final T-duality axiom, assuming $(X^\sharp, H^\sharp)$ is T-dual to $(X, 0)$. Quite magically, this turns out to work!

Automorphisms of CT-Algebras

- Actually, there is an advantage to working even more generally, since if the H-flux on X is non-zero, we will need an action of $\mathbb{R}$ on the stable CT-algebra with spectrum X and Dixmier-Douady invariant H, since the K-theory of this algebra is $K^*(X, H)$.
- So we need to understand the structure of $\operatorname{Aut} CT(X, H)$ (as a topological group) and actions of $\mathbb{R}$ on $CT(X, H)$.
- Every $*$-automorphism of $\mathcal{K}(\mathcal{H})$ is induced by a unitary, so $\operatorname{Aut} \mathcal{K} \cong PU(\mathcal{H})$.
- There is an obvious map $\sigma\colon\ \operatorname{Aut} CT(X, H) \to \operatorname{Homeo}(X)$ gotten by sending an automorphism to the induced action on the spectrum.

The Phillips-Raeburn Theorem

Theorem (Phillips-Raeburn)

Let A be a stable CT-algebra with spectrum X and Dixmier-Douady invariant H. Then the image of the map $\sigma\colon\ \operatorname{Aut} CT(X,H) \to \operatorname{Homeo}(X)$ is precisely the stabilizer of $H \in H^3(X,\mathbb{Z})$. Let $\operatorname{Aut}_X CT(X,H) = \ker\sigma$. Then there is a natural short exact sequence

$$1 \to \operatorname{Inn} CT(X,H) \to \operatorname{Aut}_X CT(X,H) \xrightarrow{\rho} H^2(X,\mathbb{Z}) \to 0.$$

Here $\operatorname{Inn} CT(X,H)$ is the group of automorphisms implemented by unitary multipliers. These are precisely the automorphisms exterior equivalent to the identity.

Actions on CT-Algebras

- We can think of the Phillips-Raeburn Theorem as a structure theorem for actions α of $\mathbb{Z}$ on $CT(X, H)$, modulo exterior equivalence. For example, set of equivalence classes of actions of $\mathbb{Z}$ on $C_0(X, \mathcal{K})$ can be identified with

$$H^2(X, \mathbb{Z}) \rtimes \text{Homeo}(X).$$

- The $H^2(X, \mathbb{Z})$ arises as $[X, \text{Aut}\,\mathcal{K} = PU \simeq K(\mathbb{Z}, 2)]$. $\rho(\alpha)$ can be identified with the Chern class of the circle bundle $\big(CT(X, H) \rtimes_\alpha \mathbb{Z}\big)\widehat{\ } \to X$.
- For our purposes we need a similar structure theorem for actions of $\mathbb{R}$ on $CT(X, H)$, modulo exterior equivalence.

Actions of $\mathbb{R}$ on CT-Algebras

Lemma (Lifting)

Suppose one is given an action α of $\mathbb{R}$ on a (2nd countable) locally compact space X (of the homotopy type of a finite CW complex). Let A be a stable CT-algebra with spectrum X. Then α lifts to an action $\widetilde{\alpha}$ of $\mathbb{R}$ on A, and the lifted action is unique up to exterior equivalence.

Idea of Proof.

A corresponds to a locally trivial principal PU-bundle over X. So the problem is to lift an action on the base of such a bundle to an action on the bundle. Suppose for simplicity that X, the bundle, and the $\mathbb{R}$-action are smooth. Then we need to lift a vector field on the base to a "horizontal" vector field on the bundle, and a choice of a connection is precisely what is needed for this. Analysis of the number of ways to do this gives the uniqueness statement. □

The Raeburn-Rosenberg Theorem

Theorem (Raeburn-Rosenberg)

Let X be a (2nd countable) locally compact space (of the homotopy type of a finite CW complex). Suppose X is the total space of a principal S^1-bundle $p\colon X \to Z$, and suppose $H \in H^3(X, \mathbb{Z})$. Let $A = CT(X, H)$. Lift the free action of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ on X to a locally free action of $\mathbb{R}$ on X, and then to an action α of $\mathbb{R}$ on A using the Lifting Lemma. Let $B = A \rtimes_\alpha \mathbb{R}$. Then B is also a stable CT-algebra, with spectrum $X^\sharp$ the total space of another principal S^1-bundle $p^\sharp\colon X^\sharp \to Z$. The bundles and Dixmier-Douady classes are related by

$$c_1(p) = (p^\sharp)_!(H^\sharp) \quad \text{and} \quad c_1(p^\sharp) = (p)_!(H). \tag{16}$$

The algebra $B \rtimes_{\widehat{\alpha}} \mathbb{R}$ is isomorphic to A again.

The Raeburn-Rosenberg Theorem (cont'd)

Theorem (Raeburn-Rosenberg)

In fact, if Y is the spectrum of $A \rtimes_{\alpha|_{\mathbb{Z}}} \mathbb{Z}$, or equivalently, of $B \rtimes_{\widehat{\alpha}|_{\mathbb{Z}}} \mathbb{Z}$, there is a commuting diagram of principal S^1-bundles

$$\begin{array}{ccccc} & & Y & & \\ & \swarrow^{(p^\sharp)^* p} & & \searrow^{p^*(p^\sharp)} & \\ X^\sharp & & & & X \\ & \searrow_{p^\sharp} & & \swarrow_{p} & \\ & & Z & & \end{array} \quad . \tag{17}$$

An Approach to Higher-Dimensional T-Duality

Next we consider T-duality to the case of spacetimes X "compactified on a higher-dimensional torus," using the C^*-algebraic method discussed above. Again we start with a principal $\mathbb{T}^n$-bundle $p\colon X \to Z$ and an "H-flux" $H \in H^3(X, \mathbb{Z})$. We need to assume that H is trivial when restricted to each $\mathbb{T}^n$-fiber of p. This of course is no restriction if $n = 2$. Proceeding as before, we want to lift the free action of $\mathbb{T}^n$ on X to an action on the continuous-trace algebra $A = CT(X, H)$. Usually there is no hope to get such a lifting for $\mathbb{T}^n$ itself, so we go to the universal covering group $\mathbb{R}^n$. If $\mathbb{R}^n$ acts on A so that the induced action on $\widehat{A}$ is trivial on $\mathbb{Z}^n$ and factors to the given action of $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, then we can take the crossed product $A \rtimes \mathbb{R}^n$ and use Connes' Thom Isomorphism Theorem to get an isomorphism between $K^{*+n}(X, H)$ and $K_*(A \rtimes \mathbb{R}^n)$.

Recovering Topological T-Duality

Under favorable circumstances, we can hope that the crossed product $A \rtimes \mathbb{R}^n$ will again be a continuous-trace algebra $CT(X^\sharp, H^\sharp)$, with $p^\sharp \colon X^\sharp \to Z$ a new principal $\mathbb{T}^n$-bundle and with $H^\sharp \in H^3(X^\sharp, \mathbb{Z})$. If we then act on $CT(X^\sharp, H^\sharp)$ with the dual action of $\widehat{\mathbb{R}}^n$, then by Takai Duality and stability, we come back to where we started. So we have a topological T-duality between (X, H) and $(X^\sharp, H^\sharp)$. Furthermore, we have an isomorphism

$$K^{*+n}(X, H) \cong K^*(X^\sharp, H^\sharp),$$

as required for matching of D-brane charges under T-duality. Two new problems now arise: potential non-uniqueness of the T-dual and "missing" T-duals. These can be explained either by non-uniqueness of the lift to an action of $\mathbb{R}^n$ on $A = CT(X, H)$, or by failure of the crossed product to be a continuous-trace algebra.

A Crucial Example

Let's now examine what happens when we try to carry out this program in one of our "problem cases," $n = 2$, $Z = S^1$, $X = T^3$ (a trivial $\mathbb{T}^2$-bundle over S^1), and H the usual generator of $H^3(T^3)$. First we show that there is an action of $\mathbb{R}^2$ on $CT(X, H)$ compatible with the free action of $\mathbb{T}^2$ on X with quotient S^1. We will need the notion of an induced action. We start with an action α of $\mathbb{Z}^2$ on $C(S^1, \mathcal{K})$ which is trivial on the spectrum. This is given by a map $\mathbb{Z}^2 \to C(S^1, \operatorname{Aut} \mathcal{K}) = C(S^1, PU(L^2(\mathbb{T})))$ sending the two generators of $\mathbb{Z}^2$ to the maps

$$w \mapsto \text{multiplication by } z,$$
$$w \mapsto \text{translation by } w.$$

(These unitaries commute in PU, not in U.)

A Calculation

Now form $A = \operatorname{Ind}_{\mathbb{Z}^2}^{\mathbb{R}^2} C(S^1, \mathcal{K})$. This is a C^*-algebra with $\mathbb{R}^2$-action $\operatorname{Ind}\alpha$ whose spectrum (as an $\mathbb{R}^2$-space) is $\operatorname{Ind}_{\mathbb{Z}^2}^{\mathbb{R}^2} S^1 = S^1 \times \mathbb{T}^2 = X$. We can see that $A \cong CT(X, H)$ via "inducing in stages". Let $B = \operatorname{Ind}_{\mathbb{Z}}^{\mathbb{R}} C(S^1, \mathcal{K}(L^2(\mathbb{T})))$ be the result of inducing over the first copy of $\mathbb{R}$. It's clear that $B \cong C(S^1 \times \mathbb{T}, \mathcal{K})$. We still have another action of $\mathbb{Z}$ on B coming from the second generator of $\mathbb{Z}^2$, and $A = \operatorname{Ind}_{\mathbb{Z}}^{\mathbb{R}} B$. The action of $\mathbb{Z}$ on B is by means of a map $\sigma \colon S^1 \times \mathbb{T} \to PU(L^2(\mathbb{T})) = K(\mathbb{Z}, 2)$, whose value at (w, z) is the product of multiplication by z with translation by w. Thus A is a CT-algebra with Dixmier-Douady invariant $[\sigma] \times c = H$, where $[\sigma] \in H^2(S^1 \times \mathbb{T}, \mathbb{Z})$ is the homotopy class of σ and c is the usual generator of $H^1(S^1, \mathbb{Z})$.

A Calculation (cont'd)

Now that we have an action of $\mathbb{R}^2$ on $A = CT(X, H)$ inducing the free $\mathbb{T}^2$-action on the spectrum X, we can compute the crossed product to see what the associated "T-dual" is. Since $A = \mathrm{Ind}_{\mathbb{Z}^2}^{\mathbb{R}^2} C(S^1, \mathcal{K})$, we can use the Green Imprimitivity Theorem to see that

$$A \rtimes_{\mathrm{Ind}\,\alpha} \mathbb{R}^2 \cong \left(C(S^1, \mathcal{K}) \rtimes_\alpha \mathbb{Z}^2 \right) \otimes \mathcal{K}.$$

The rotation algebra A_θ is the universal C^*-algebra generated by unitaries U and V with $UV = e^{2\pi i\theta} VU$. So if we look at the definition of α, we see that $A \rtimes_{\mathrm{Ind}\,\alpha} \mathbb{R}^2$ is the algebra of sections of a bundle of algebras over S^1, whose fiber over $e^{2\pi i\theta}$ is $A_\theta \otimes \mathcal{K}$. Alternatively, it is Morita equivalent to $C^*(\Gamma)$, where Γ is the discrete Heisenberg group of strictly upper-triangular 3×3 integral matrices.

Noncommutative T-Duals

Put another way, we could argue that we've shown that $C^*(\Gamma)$ is a noncommutative T-dual to (T^3, H), both viewed as fibering over S^1. So we have an explanation for the missing T-dual: we couldn't find it just in the world of topology alone because it's noncommutative.

Part VII

Problems Presented by S-Duality

Review of S-Duality

Recall from Lecture 1 that S-duality is supposed to be in conformity with the Montonen-Olive conjecture about an exact quantum electric-magnetic duality. While it is only conjectural, there is a fair amount of evidence for it. S-duality applies in several different situations, but the most important are a self-duality of type IIB theory and a duality between type I and *SO* heterotic theory. In this lecture we will discuss some problems these dualities present, and possible solutions in some cases.

S-Duality of Type IIB

First let's discuss the type IIB theory on a spacetime X. In this theory, D-branes are even dimensional and D-brane charges live in $K(X)$. There are also Ramond-Ramond (RR) fields C_0, C_2, and C_4 whose field strengths are odd dimensional closed forms G_1, G_3 and G_5 of degrees 1, 3, and 5, with G_5 self-dual. The associated charges live either in odd de Rham cohomology or odd K-theory $K^{-1}(X)$. Recall that we also have the Neveu-Schwarz H-flux H, the field strength of the B-field.

S-duality is supposed to be an $SL(2, \mathbb{Z})$ symmetry that mixes "electric" and "magnetic" charges, examples of which come from the fundamental string and the D-string which couple to B and C_2, respectively.

Puzzle of the 3-Cohomology Classes

Now it makes sense for $SL(2, \mathbb{Z})$ to act on $H^3(X, \mathbb{Z}) \times H^3(X, \mathbb{Z})$, and under this action everything in the orbit of the class $([H], [G_3])$ is a charge in an S-dual theory. In other words, from the point of view of S-duality, H and G_3 play symmetrical roles. However, there is a peculiarity here: charges are really supposed to live in twisted K-theory, not de Rham cohomology, and the twist is given by H, not G_3, so there is an inherent asymmetry here. This issue was studied by Bouwknegt, Evslin, Jurčo, Mathai, and Sati, but hasn't been totally settled yet.

Conjectured Dualities

As we mentioned before, there is believed to be an S-duality relating type I string theory to one of the heterotic string theories. There are also various other dualities relating these two theories to type IIA theory. Putting these together, we expect a (non-perturbative) duality between type I string theory on $T^4 \times \mathbb{R}^6$ and type IIA theory on $K3 \times \mathbb{R}^6$, at least at certain points in the moduli space.

How can we reconcile this with the principle that brane charges in type I should take their values in KO, while brane charges in type IIA should take their values in K^{-1}?

On the face of it, this appears ridiculous: $KO(T^4 \times \mathbb{R}^6) = KO^{-6}(T^4)$ has lots of 2-torsion, while $K^*(K3)$ is all torsion-free and concentrated in even degree.

KO-Theory of T^4

One side is easy compute. Recall that for any space X,

$$KO^{-j}(X \times S^1) \cong KO^{-j}(X) \oplus KO^{-j-1}(X).$$

Iterating, we get

$$\begin{aligned} KO^{-6}(T^4) &\cong KO^{-6} \oplus 4KO^{-7} \oplus 6KO^{-8} \oplus 4KO^{-9} \oplus KO^{-10} \\ &\cong \mathbb{Z}^6 \oplus (\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/2) \cong \mathbb{Z}^6 \oplus (\mathbb{Z}/2)^5. \end{aligned}$$

K-Theory of the Orbifold Limit of K3

The way we deal with the opposite side of the duality is to recall that a K3 surface can be obtained by blowing up the point singularities in T^4/G, where $G = \mathbb{Z}/2$ acting by multiplication by -1 on $\mathbb{R}^4/\mathbb{Z}^4$. This action is semi-free with 16 fixed points, the points with all four coordinates equal to 0 or $\frac{1}{2}$ mod $\mathbb{Z}$. If fact one way of deriving the (type I on T^4) $\leftrightarrow$ (type IIA on $K3$) duality explicitly uses the orbifold T^4/G.

But what group should orbifold brane charges live in? Not just $K^*(T^4/G)$, as this ignores the orbifold structure. One solution that has been proposed is $K^*_G(T^4)$, which we computed. However, as we'll see, there appears to be a better candidate.

Cohomology Calculations

Let M be the result of removing an open ball around each G-fixed point in T^4. This is a compact manifold with boundary on which G acts freely; let $N = M/G$. We get a K3 surface back from N by gluing in 16 copies of the unit disk bundle of the tangent bundle of S^2 (known to physicists as the Eguchi-Hanson space), one along each $\mathbb{RP}^3$ boundary component in ∂N.

Theorem (with S. Mendez-Diez)

$$H^i(N, \partial N) \cong H_{4-i}(N) \cong \begin{cases} 0, & i = 0 \\ \mathbb{Z}^{15}, & i = 1 \\ \mathbb{Z}^6, & i = 2 \\ (\mathbb{Z}/2)^5, & i = 3 \\ \mathbb{Z}, & i = 4 \\ 0, & \textit{otherwise.} \end{cases}$$

K-Theory Calculations

Recall N is the manifold with boundary obtained from T^4/G by removing an open cone neighborhood of each singular point.

Theorem (with S. Mendez-Diez)

$K^0(N, \partial N) \cong K_0(N) \cong \mathbb{Z}^7$ and $K^{-1}(N, \partial N) \cong K_1(N) \cong \mathbb{Z}^{15} \oplus (\mathbb{Z}/2)^5$.

Note that the reduced K-theory of (T^4/G) mod (singular points) is the same as $K^*(N, \partial N)$. Note the resemblance of $K^{-1}(N, \partial N)$ to $KO^{-6}(T^4) \cong \mathbb{Z}^6 \oplus (\mathbb{Z}/2)^5$. While they are not the same, the calculation suggests that the brane charges in type I string theory on $T^4 \times \mathbb{R}^6$ do indeed show up some way in type IIA string theory on the orbifold limit of $K3$.

Equivariant K-Theory Calculations

Again let $G = \mathbb{Z}/2$. Equivariant K-theory K_G^* is a module over the representation ring $R = R(G) = \mathbb{Z}[t]/(t^2 - 1)$. This ring has two important prime ideals, $I = (t - 1)$ and $J = (t + 1)$. We have $R/I \cong R/J \cong \mathbb{Z}$, $I \cdot J = 0$, $I + J = (I, 2) = (J, 2)$, $R/(I + J) = \mathbb{Z}/2$.

Theorem (with S. Mendez-Diez)

$K_G^0(\mathbb{T}^4) \cong R^8 \oplus (R/J)^8$, and $K_G^{-1}(\mathbb{T}^4) = 0$. Also, $K_G^0(M, \partial M) \cong (R/I)^7$, $K_G^{-1}(M, \partial M) \cong (R/I)^{10} \oplus (R/2I)^5$.

Discussion

Note that the equivariant K-theory calculation is a refinement of the ordinary K-theory calculation (since G acts freely on M and ∂M with quotients N and ∂N, so that $K_G^*(M)$ and $K_G^*(\partial M)$ are the same as $K^*(N)$ and $K^*(\partial N)$ *as abelian groups*, though with the addition of more structure). While we don't immediately need the extra structure, it may prove useful later in matching brane charges from $KO(T^4 \times \mathbb{R}^6)$ on specific classes of branes.

Other Cases of Type I/Type II Charge Matching

More generally, one could ask if there are circumstances where understanding of K-theory leads us to expect the possibility of a string duality between type I string theory on a spacetime Y and type II string theory on a spacetime Y'. For definiteness, we will assume we are dealing with type IIB on Y'. (This is no great loss of generality since as seen in the last lecture, types IIA and IIB are related via T-duality.) Matching of stable brane D-charges then leads us to look for an isomorphism of the form

$$KO^*(Y) \cong K^*(Y').$$

In general, such isomorphisms are quite rare, in part because of 2-torsion in KO^{-1} and KO^{-2}, and in part because KO-theory is usually 8-periodic rather than 2-periodic.

A Conjectural Duality

But there is one notable exception: one knows that

$$KO \wedge (S^0 \cup_\eta e^2) \simeq K,$$

where $S^0 \cup_\eta e^2$ is the stable cell complex obtained by attaching a stable 2-cell via the stable 1-stem η. This is stably the same (up to a degree shift) as $\mathbb{CP}^2$, since the attaching map $S^3 \to S^2 \cong \mathbb{CP}^1$ for the top cell of $\mathbb{CP}^2$ is the Hopf map, whose stable homotopy class is η. Thus one might expect a duality between type I string theory on $X^6 \times \left(\mathbb{CP}^2 \smallsetminus \{\text{pt}\}\right)$ and type IIB string theory on $X^6 \times \mathbb{R}^4$. We plan to look for evidence for this.

Part VIII

The AdS/CFT Correspondence and Problems it Raises

Maldacena's Idea

The AdS/CFT correspondence or holographic duality is a conjectured physical duality, proposed by Juan Maldacena, of a different sort, relating IIB string theory on a 10-dimensional spacetime manifold to a gauge theory on another space. In the original version of this duality, the string theory lives on $AdS^5 \times S^5$, and the gauge theory is $\mathcal{N} = 4$ super-Yang-Mills theory on Minkowski space $\mathbb{R}^{1,3}$. Other versions involve slightly different spaces and gauge theories. Notation:

- $\mathcal{N}$ is the standard notation for the supersymmetry multiplicity. In other words, $\mathcal{N} = 4$ means there are 4 sets of supercharges, and there is a $U(4)$ R-symmetry group acting on them.
- AdS^5 is (up to coverings) $SO(4,2)/SO(4,1)$. Topologically, it's $\mathbb{R}^4 \times S^1$. It's better to pass to the universal cover, so that time isn't periodic.

Nature of the Correspondence

We have already explained that D-branes carry Chan-Paton bundles. In type IIB string theory, a collection of N coincident D3 branes have $3 + 1 = 4$ dimensions and carry a $U(N)$ gauge theory living on the Chan-Paton bundle. This gauge theory is the holographic dual of the string theory, and the number N can be recovered as the flux of the Ramond-Ramond (RR) 5-field G_5 through S^5. The rotation group $SO(6)$ of $\mathbb{R}^5$ is identified with the $SU(4)_R$ symmetry group of the $\mathcal{N} = 4$ gauge theory.

The AdS/CFT correspondence looks like holography in that physics in the bulk of AdS space is described by a theory of one less dimension "on the boundary." This can be explained by the famous relation between the entropy of a black hole and the area of its boundary, which in turn forces quantum gravity theories to obey a holographic principle.

The Lagrangian for 4D SYM

Recall that the Montonen-Olive Conjecture asserts that classical electro-magnetic duality should extend to an exact symmetry of certain quantum field theories. 4-dimensional super-Yang-Mills (SYM) with $\mathcal{N} = 4$ supersymmetry is believed to be a case for which this conjecture applies. The Lagrangian involves the usual Yang-Mills term

$$\frac{-1}{4g_{YM}^2} \int \mathrm{Tr}(F \wedge *F)$$

and the theta angle term (related to the Pontrjagin number or instanton number)

$$\frac{\theta}{32\pi^2} \int \mathrm{Tr}(F \wedge F).$$

We combine these by introducing the tau parameter

$$\tau = \frac{4\pi i}{g_{YM}^2} + \frac{\theta}{2\pi}.$$

Charges in 4D SYM

The tau parameter measures the relative size of "magnetic" and "electric charges." Dyons (particles with mixed electric/magnetic charge) in SYM have charges (m, n) living in the group $\mathbb{Z}^2$; the associated *complex charge* is $q + ig = q_0(m + n\tau)$. The electro-magnetic duality group $SL(2, \mathbb{Z})$ acts on τ by linear fractional transformations. More precisely, it is generated by two transformations: $T\colon\ \tau \mapsto \tau + 1$, which just increases the θ-angle by 2π, and has no effect on magnetic charges, and by $S\colon\ \tau \mapsto -\frac{1}{\tau}$, which effectively interchanges electric and magnetic charge. By the Montonen-Olive conjecture, the same group $SL(2, \mathbb{Z})$ should operate on type IIB string theory in a similar way, and θ should correspond in the string theory to the expectation value of the RR scalar field C_0.

$AdS^5 \times S^5$ and SYM

Let $X = \mathbb{R}^5 \times S^5$, where $\mathbb{R}^5$ is the universal cover of AdS^5. We think of $\mathbb{R}^5$ more exactly as $\mathbb{R}^4 \times \mathbb{R}_+$, so that $\mathbb{R}^4 \times \{0\}$, Minkowski space, is "at the boundary." Then we want to compare the K-theoretic charge groups for type IIB string theory on X and for $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$. The RR field charges should live in $K^{-1}(X)$, but we see this requires clarification: the RR field G_5 should represent the number N in $H^5(S^5)$, so we need to use homotopy theoretic K-theory K_h here instead of K-theory with compact support, which we've implicitly been using before. Indeed, note that
$K^{-1}(X) \cong K^{-1}(\mathbb{R}^5) \otimes K^0(S^5) \cong H^0(S^5)$, while
$K_h^{-1}(X) \cong K_h^0(\mathbb{R}^5) \otimes K^{-1}(S^5) \cong H^5(S^5)$, which is what we want.

Now what about the D-brane charge group for the string theory? This should be $\mathbb{Z} \cong K^0(X) \cong K^0(\mathbb{R}^4 \times Y) \cong K^0(\mathbb{R}^4) \otimes K^0(Y)$, where Y is the D5-brane $\mathbb{R} \times S^5$, which has $K^0(Y) \cong \mathbb{Z}$. Note that this is naturally isomorphic to $K^0(\mathbb{R}^4) = \widehat{K}^0(S^4)$, which is where the instanton number lives in the dual gauge theory. But what charge group on X corresponds to the group of electric and magnetic charges in the gauge theory?

Lens Spaces

It is believed that the string/gauge correspondence should apply much more generally, to many type IIB string theories on spaces other than $AdS^5 \times S^5$, and to gauge theories with less supersymmetry than the $\mathcal{N} = 4$ theory that we've been considering. In particular, given a representation of a cyclic group $\mathbb{Z}/k \to SU(3)$, say with k an odd prime, $\mathbb{Z}/k$ acts on $S^5 \subset \mathbb{C}^3$, without fixed points unless the representation contains the trivial representation, and the quotient is a lens space L^5. Many different lens spaces can arise, even with the same k! And it is believed [Morrison-Plesser] that one has an AdS/CFT correspondence between type IIB string theory on $AdS^5 \times L^5$ and an $\mathcal{N} = 1$ SYM theory on $\mathbb{R}^4$, this time for the group $U(N)^k$. The rank nk of the gauge group can be explained since $\langle [G_5], [L^5] \rangle = N$, so Nk is the pairing of (the lift of) G_5 with $[S^5]$, S^5 the k-fold cover of L^5.

K-Theory for Lens Spaces

For a lens space L^5 with fundamental group $\mathbb{Z}/k$, the K-groups are $K^{-1}(L) \cong \mathbb{Z}$ and $K^0(L) \cong \mathbb{Z} \oplus T$, where the torsion group T has order k^2 but may or may not be cyclic, depending on k and the specific lens space chosen. As in the last example, the class $[G_5]$ should live in a group identifiable with $H^5(L)$ or $K^{-1}(L)$, which is naturally isomorphic to $K_h^{-1}(\mathbb{R}^5 \times L^5)$, not to $K^{-1}(\mathbb{R}^5 \times L^5)$ (with compact support). So we're led to a question: when the spacetime has the form $\mathbb{R}^4 \times Y$, should the D-brane charges should really live in $K^0(\mathbb{R}^4 \times Y) \cong K^0(Y)$, or maybe in $K_h^0(Y)$?

K-Theory for Lens Spaces (cont'd)

In our case, $Y = \mathbb{R} \times L^5$ is noncompact (unlike the more typical case where Y is a compact Calabi-Yau 3-fold). So $K^0(Y) \cong K^{-1}(L)$ doesn't see the torsion in $K^*(L)$, but $K_h^0(Y) \cong K^0(L)$ does. So one is led to an interesting physics question: in this example, do we expect to see D-brane charge groups with torsion? If so, they should also show up in the dual conformal gauge theory. There is a possibility, hinted at in the work of Morrison-Plesser, that this could happen because of symmetry breaking and an anomaly cancellation condition. If so, then the distinction between the cases $k = 3$ (and the standard lens space coming from the action of $e^{2\pi i/3}$ by scalar multiplication) and $k \geq 5$ (where the K-group is cyclic) is especially interesting.

Part IX

KR-Theory and some *KR* Calculations

Real Spaces and Bundles

Definition (Atiyah)

A Real space is a locally compact space X with an involution $\iota\colon X \to X$ (so $\iota^2 = 1$). A Real vector bundle on such a space is a complex vector bundle $E \xrightarrow{p} X$ with a conjugate-linear involution $\bar{\ }$ such that $\bar{\ }\colon E_x \to E_{\iota(x)}$.

Example

- Any locally compact space X is a Real space with the trivial involution. A Real vector bundle on X is then the complexification of a real vector bundle, with $\bar{\ }$ = complex conjugation. $\mathbb{R}$ with the trivial involution will be denoted $\mathbb{R}^{1,0}$.
- The real line $\mathbb{R}$ is a Real space under the involution $\iota(x) = -x$. This space is hereafter denoted $\mathbb{R}^{0,1}$. We let $\mathbb{R}^{p,q} = \left(\mathbb{R}^{1,0}\right)^p \oplus \left(\mathbb{R}^{0,1}\right)^q$.

KR-Theory

Definition (Atiyah)

If (X, ι) is a compact Real space, $KR(X)$ denotes the Grothendieck group of equivalence classes of Real vector bundles on X. Note that if ι is trivial, this is just $KO(X)$. KR is extended to locally compact spaces as usual: let X^+ be the one-point compactification, with ι extended to fix the point at infinity, and let $KR(X) = \ker\left(KR(X^+) \to KO(\mathrm{pt})\right)$. The usual proof shows that this extends to a cohomology theory KR^*, with $KR^{-j}(X) = KO(X \times \mathbb{R}^{j,0})$.

Theorem (Atiyah)

$KR(\mathbb{R}^{1,1}) = \widetilde{KR}(\mathbb{CP}^1) \cong \mathbb{Z}$, with generator the usual Bott element β. (Here the involution on $\mathbb{CP}^1$ is complex conjugation.)

Bott Periodicity

In fact for any Real space X, $KR(X) \cong KR(X \times \mathbb{R}^{1,1})$ via external product with β. Thus we can define $KR^{p,q}(X) = KR(X \times \mathbb{R}^{p,q})$, and this only depends on $p - q$. In other words, we can think of $\mathbb{R}^{0,1}$ as $\mathbb{R}^{-1}$! Since $KR(\mathbb{R}^{p,0}) \cong KO^p$, the groups $KR^*(X)$ are periodic with period 8. One other case deserves special mention. If X is any locally compact space and with give $X \amalg X$ the involution interchanging the two factors, then any complex vector bundle on X extends uniquely to a Real vector bundle on $X \amalg X$. Thus $KR^*(X \amalg X, \circlearrowright) \cong K^*(X)$. Let $S^{p,q}$ be the unit sphere in $\mathbb{R}^{p,q}$. Note that this is topologically S^{p+q-1} (with an involution). $S^{0,2}$ is S^1 with the antipodal map, and $S^{1,1}$ is the one-point compactification of $\mathbb{R}^{0,1}$.

Special cases

Theorem (Karoubi-Weibel)

If the involution ι on X is free, then $KR^*(X)$ is periodic of period 4.

Theorem (Atiyah)

For any Real space X, $KR^*(X \times S^{0,1}) \cong K^*(X)$, and $KR^*(X \times S^{0,2}) \cong KSC^*(X)$, the self-conjugate K-theory of Donald Anderson and Paul Green. This has coefficient groups

$$KSC^{-j} \cong \begin{cases} \mathbb{Z}, & j \equiv 0 \text{ or } 3 \mod 4, \\ \mathbb{Z}/2, & j \equiv 1 \mod 4, \\ 0, & j \equiv 2 \mod 4. \end{cases}$$

Holomorphic Involutions

In the last lecture, we will be especially interested in the case where X is a 2-torus with a complex structure, that is, an elliptic curve, and ι is either a holomorphic or an antiholomorphic involution. We start with the case of holomorphic involutions. We can assume $X = \mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$, $\tau \in \mathfrak{h}$ (the upper half-plane).

Theorem (Classical)

Let $X = \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. If ι is a holomorphic involution of X and ι is not the identity, then either $\iota(z) = z + \delta$, with $2\delta \in \Lambda$, or else $\iota(z) = -z + \delta$, with δ arbitrary (in a fundamental domain). In the first case, there are no fixed points. In the second case, $X^\iota = \{z \in \mathbb{C} : z \equiv \delta/2 \mod \Lambda\}/\Lambda$ consists of 4 points.

Antiholomorphic Involutions

Theorem (Classical)

Let $X = \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. If ι is an antiholomorphic involution of X, then ι is of the form $z \mapsto \alpha\bar{z} + \beta$ and X^ι is a disjoint union of either 0, 1, or 2 circles. In these three cases, X/ι is topologically a Klein bottle, a Möbius strip, or an annulus. An elliptic curve X admits an antiholomorphic involution if and only if the j-invariant of X is real.

Remark: The j-invariant does not by itself determine the antiholomorphic involution. The moduli space of these involutions has two connected components.

Topological Types

We can summarize the topological types of involutions on elliptic curves as follows:

- Each holomorphic involution of an elliptic curve X is topologically conjugate to exactly one of the following: the identity, $S^{0,2} \times S^{2,0}$ (free action), or $S^{1,1} \times S^{1,1}$ (four fixed points).
- Each antiholomorphic involution of an elliptic curve X is topologically conjugate to exactly one of the following: $S^{1,1} \times S^{2,0}$ (species 2); $S^{1,1} \times S^{0,2}$ (species 0); or an involution with one fixed circle where the action on the complement of this circle is free and orientation-reversing, with quotient an open Möbius strip (species 1).

Connections with Real Algebraic Geometry

There is an interesting connection between antiholomorphic involutions on elliptic curves and real algebraic geometry. An elliptic curve together with an antiholomorphic involution ι is equivalent to an elliptic curve (a smooth projective curve of genus 1) defined over $\mathbb{R}$. The number of connected components of the set of real points is called the species s.

Theorem (Harnack, 1876)

Let C be a smooth curve of genus g defined over $\mathbb{R}$. Then the set of real points $C(\mathbb{R})$ is a disjoint union of s circles, where $0 \leq s \leq g+1$. Any of these values can occur.

This of course explains why we have 3 topological types of antiholomorphic involutions.

The Karoubi-Weibel Theorem

Karoubi and Weibel [2003] observed that if X is a quasi-projective variety defined over $\mathbb{R}$, then $X(\mathbb{C})$, together with the involution ι defined by complex conjugation, becomes a Real space, and thus one gets natural maps $K_n(X) \to KR^{-n}(X(\mathbb{C}), \iota)$. Here $K_n(X)$ is algebraic K-theory.

Theorem (Karoubi-Weibel)

If X is a smooth variety defined over $\mathbb{R}$, then $K_n(X; \mathbb{Z}/2^\nu) \to KR^{-n}(X(\mathbb{C}), \iota; \mathbb{Z}/2^\nu)$ is an isomorphism on K-theory with coefficients in $\mathbb{Z}/2^\nu$ for $n \geq \dim X$.

Theorem (Pedrini-Weibel)

If X is a smooth curve defined over $\mathbb{R}$, then $K_n(X)$ is the direct sum of a divisible group and an elementary abelian 2-group for $n \neq 0$, and the torsion in $K_n(X)$ is 8-periodic (4-periodic if $X(\mathbb{R}) = \emptyset$) and explicitly computed.

The Basic Problem

For applications which will come in the last lecture, we now want to compute the KR groups for all Real spaces which can be made out of elliptic curves X with a holomorphic or antiholomorphic involution. It's convenient to add the cases of $(X \amalg X, (z, w) \leftrightarrow (w, z))$ and $(X \amalg \bar{X}, (z, \bar{w}) \leftrightarrow (w, \bar{z}))$. We only need to consider one representative for each topological type.

Holomorphic Involutions	
Real Space	**K-Theory**
$X \amalg X$	$K^*(T^2)$
$S^{2,0} \times S^{2,0}$	$KO^*(T^2)$
$S^{0,2} \times S^{2,0}$	$KSC^*(S^1) \cong KSC^* \oplus KSC^{*-1}$
$S^{1,1} \times S^{1,1}$	$KO^{*+2}(T^2)$

The Antiholomorphic Case

We can make a similar table for the antiholomorphic involutions.

Antiholomorphic Involutions		
Real Space	**Species**	***K*-Theory**
$X \amalg \bar{X}$		$K^*(T^2)$
$S^{1,1} \times S^{0,2}$	0	$KSC^*(S^{1,1}) \cong KSC^* \oplus KSC^{*+1}$
$S^{1,1} \times S^{2,0}$	2	$KO^{*+1}(T^2)$
	1	complicated, see below

The Hard Case: Species 1

This analysis leaves us with one hard case that can't be done by "inspection," the case of antiholomorphic involutions of species 1. The reason is that there is no involution on S^1 with only one fixed point, so there is no way to decompose a species 1 involution as a product of two 1-dimensional Real spaces.

Lemma

Let X be T^2 with an orientation-reversing involution ι of species 1. Then there is a long exact sequence

$$\cdots \to KO^{j-2} \xrightarrow{\delta} KSC^{j+1} \to \widetilde{KR}^j(X,\iota) \to KO^{j-1} \xrightarrow{\delta} KSC^{j+2} \to \cdots . \quad (18)$$

The Hard Case: Calculation

Proof.

Use the fact that $X^\iota \cong S^{2,0}$ and $X \smallsetminus X^\iota \cong S^{0,2} \times \mathbb{R}^{0,1}$. □

This does not completely determine all the groups. To finish the calculation, we need to use Karoubi-Weibel and Weibel-Pedrini applied to K-theory with $\mathbb{Z}/2$ coefficients to pin down the 2-primary torsion using the commuting diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_n(X)/2 & \longrightarrow & K_n(X;\mathbb{Z}/2) & \longrightarrow & {}_2K_{n-1}(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \cong & & \downarrow & & \\
0 & \longrightarrow & KR^{-n}(X)/2 & \longrightarrow & KR^{-n}(X;\mathbb{Z}/2) & \longrightarrow & {}_2KR^{1-n}(X) & \longrightarrow & 0.
\end{array}$$

Part X

T-Duality for Orientifolds and Applications of KR-Theory

Parity Operators

There are several parity operators that are important in string theory:

- the worldsheet parity operator Ω, that reverses orientation on the string worldsheet Σ;
- the spacetime parity operator $\mathcal{I}$, an involutive self-homeomorphism of spacetime X;
- the left worldsheet fermion parity operator $(-1)^{F_L}$, that counts the number of left-moving fermions mod 2.

The GSO (Gliozzi-Scherk-Olive) projection onto states with $(-1)^{F_L} = -1$ is designed to get rid of the tachyon, which has $(-1)^{F_L} = +1$. It has the effect of writing type I string theory as a "quotient" of type IIB theory.

Orientifolds

These parity operators suggest the idea of an orientifold theory. Basically, we require spacetime X to be equipped with an involution ι, and instead of considering all maps $\Sigma \to X$ from the worldsheet to X, we use only equivariant maps

$$(\Sigma, \Omega) \to (X, \iota).$$

We'll be especially interested in type II theories where $X = Y^{2n} \times \mathbb{R}^{10-2n}$ and (in order to preserve a lot of supersymmetry) Y is a complex manifold of complex dimension $n = 1, 2,$ or 3 with a globally non-vanishing holomorphic n-form (the Calabi-Yau condition). We assume the involution ι is trivial on the flat "physical" spacetime $\mathbb{R}^{10-2n}$ but can be nontrivial on Y. There are two main subcases:

- anti-holomorphic involutions, type IIA;
- holomorphic involutions, type IIB.

Disconnected Orientifolds

There is one subcase of this framework where spacetime has two connected components, and we replace Y by either $Y \amalg Y$ with involution $(z, w) \leftrightarrow (w, z)$ or by $Y \amalg \bar{Y}$ with involution $(z, \bar{w}) \leftrightarrow (w, \bar{z})$. These cases are actually equivalent to traditional type IIA and type IIB string theory on Y, since an equivariant map from the worldsheet requires the worldsheet also to be disconnected, and we might as well look at nonequivariant maps $\Sigma \to Y$ instead of equivariant maps $(\Sigma \amalg \Sigma) \to (Y \amalg Y)$, etc. Thus the orientifold formalism recovers traditional string theory as a special case.

Brane Charges in Orientifold Theories

Just as D-brane charges in usual string theory take their values in K-theory, the same is true for orientifold theories, except that we have to take the involution ι into account. The obvious candidate for doing this is $KR^*(X, \iota)$. This is reasonable since the bundles we want to consider should pull back to bundles on Σ with a conjugate-linear action compatible with parity reversal Ω.

Note that we have a few interesting special cases. If ι is trivial, that means our strings really factor through Σ/Ω, i.e., we consider unoriented strings. This is the type I theory and as we expect, the brane charges live in $KR^*(X, \mathrm{id}) = KO^*(X)$.

In the case where X has two components and ι interchanges them, we recover type IIA or type IIB string theory, and charges live in $KR^*(X \amalg X) \cong K^*(X)$, as expected.

Elliptic Curve Orientifolds

All the material from now on is joint work with Chuck Doran and Stefan Mendez-Diez. We will be studying orbifold string theories on $Y^2 \times \mathbb{R}^8$, where Y is a torus equipped with a complex structure, that is an elliptic curve (these are the only compact Calabi-Yau manifolds in complex dimension 1). We assume the involution ι is trivial on $\mathbb{R}^8$ and holomorphic or antiholomorphic on the elliptic curve. Since taking a product with $\mathbb{R}^{8,0}$ has no effect on KR because of Bott periodicity, the charge groups we are interested in are precisely the KR groups of (Y, ι), which were computed in the last lecture.

T-Duality on the Circle

Before getting to T-duality for torus orientifolds, it helps to clarify how T-duality works when the target space is S^1 with an involution. Aside from the case of $S^{2,0}$ which was studied in Lecture 1, we are interested in $S^{1,1}$ and $S^{0,2}$.

Let's suppose (this is the simplest case) that Σ, the string worldsheet, is $S^{1,1} \times \mathbb{R}$, where $\mathbb{R}$ represents time and the involution on $S^{1,1}$ is worldsheet parity reversal Ω.

T-duality is supposed to interchange winding and momentum modes in the sigma-model. The winding number for $z \mapsto z^n$ (from S^1 to S^1) is n; this mode is always equivariant when the involution is complex conjugation on both circles (the case of $S^{1,1}$), but when the target space is $S^{2,0}$, only the case $n = 0$ is equivariant, and when the target space is $S^{0,2}$, equivariance means $\bar{z}^n = z^{-n} = -z^n$, so there are no equivariant maps.

T-Duality on the Circle (cont'd)

Let's look at this in more detail. When the target space is $S^{1,1}$, if $z \in \mathbb{T}$ and $t \in \mathbb{R}$ are the coordinates on $\Sigma = S^{1,1} \times \mathbb{R}^{1,0}$ and $x\colon \Sigma \to S^{1,1}$, then quantization forces x to be periodic in t also, so x descends to the quotient space $S^{1,1} \times S^{2,0}$. But equivariance in the $S^{2,0}$ means the map is trivial in time, i.e., the momentum is 0. So maps $\Sigma \to S^{1,1}$ have arbitrary winding but vanishing momentum.

When we T-duality, winding and momentum are interchanged, so we have vanishing winding and arbitrary momentum. This is precisely the situation for $S^{2,0}$, so the orientifold targets $S^{1,1}$ and $S^{2,0}$ are T-dual to one another. This is reflected in the KR-theory: $KR^*(S^{1,1}) \cong KO^* \oplus KO^{*+1}$, while $KR^*(S^{2,0}) \cong KO^* \oplus KO^{*-1}$. These are the same up to a shift in degree by 1!

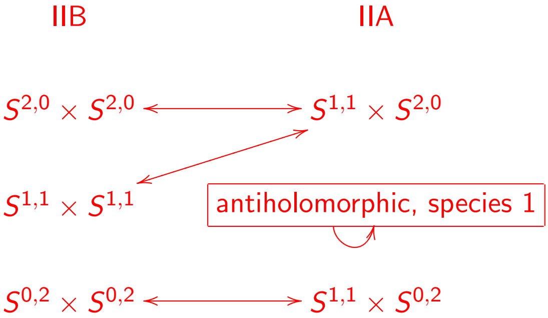
The Types of Elliptic Curve Orientifolds

Now let's go back and look at the various IIA and IIB elliptic curve orientifolds; only the topological types are relevant for the KR-theory. We have the six types

Type	Fixed Set	Real Space	KR Groups
IIB	T^2	$S^{2,0} \times S^{2,0}$	$KO^*(T^2)$
IIB	$S^0 \times S^0$	$S^{1,1} \times S^{1,1}$	$KO^{*+2}(T^2)$
IIB	$\emptyset$	$S^{2,0} \times S^{0,2}$	$KSC^*(S^1) \cong KSC^* \oplus KSC^{*-1}$
IIA	$S^1 \amalg S^1$	$S^{1,1} \times S^{2,0}$	$KO^{*+1}(T^2)$
IIA	S^1	not a product	complicated
IIA	$\emptyset$	$S^{1,1} \times S^{0,2}$	$KSC^*(S^{1,1}) \cong KSC^* \oplus KSC^{*+1}$

T-Duality of Elliptic Curve Orientifolds

Now using the T-duality between $S^{1,1}$ and $S^{2,0}$, we can see how the various orientifolds are related through T-duality. We get the following diagram of T-dualities:



Note the compatibility with the table of KR^* groups.