K-theory and Classification of Symmetric Spaces

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All new results in the following, unaccounted for, are joint with Dennis Bohle.

Symmetric Spaces

Standard definition

Definitions of ‘Symmetric Space’ vary. The probably most traditional one:

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A Riemannian manifold M is called a *Riemannian symmetric space* if for each point $x \in M$ there exists an involution s_x (a ‘symmetry’) which is an isometry of M, and has x as an isolated fixed point.

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By elementary geometry, Euclidian space, the hyperbolic plane, or the (real) spheres are examples in this class of spaces. A beautiful result going back to E. Cartan states

Theorem

A Riemannian manifold M is locally symmetric iff its curvature tensor R is parallel, $\nabla R = 0$.

More concretely

By appropriately composing symmetries, one can reach each point $m \in M$ from a given point $o \in M$ by an isometry. Symmetric spaces are hence homogeneous, and thus of the form G/K, where G is a Lie group and K is the isotropy group of a fixed point o.

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Then G/K, where K denotes the fixed point group of Σ is a (Riemannian, if K is compact) symmetric space.

Lie algebra level

Denote by $\mathfrak{g}$ the Lie algebra of G, and by σ the derivative of Σ at the unit element of G. Then

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$$

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This is the *Cartan decomposition* of $\mathfrak{g}$.

Producing a ternary product

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The space $\mathfrak{m}$ can be identified with the tangent space of G/K at the point eK. It is not a Lie subalgebra.

It has, however, the remarkable property that for any choice of elements $m_{1,2,3} \in \mathfrak{m}$,

$$[m_1, [m_2, m_3]] \in \mathfrak{m}$$

For this reason, $\mathfrak{m}$ is called a *Lie triple system*.

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(I.e. a Kähler manifold which, as a Riemannian manifold, is symmetric)

A rough classification

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This latter class can be embedded as bounded symmetric domains into $\mathbb{C}^n$, for which the role of the isometry group is taken by the group of biholomorphic automorphisms.

Connection with C*-Algebras

An example

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The action of the group $SU(n,n)$ on D_n is

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These are the biholomorphic automorphisms and look quite like Möbius transforms.

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Then D can be written as

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where $\mathrm{Iso}(\mathfrak{A})$ denotes the group of (Banach space) isometries of $\mathfrak{A}$. It coincides with the fixed points of an $\mathrm{Aut}(D)$, order 2 autmorphism Σ, defined through

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Thus, each such D is a symmetric space in a more general sense.

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Is there any relation between Lie triple and C*-products, both defined on $\mathfrak{m}$?

The answer to this question is yes.

JB*-Triple product

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$$[a,[b,c]] - i[a,[ib,c]] = \frac{ab^*c + cb^*a}{2}.$$

Definition

A closed subspace of a C*-algebra $\mathfrak{A}$ is called a *JC*-triple* iff for all $a, b, c \in \mathfrak{A}$

$$\{a,b,c\} := \frac{ab^*c + cb^*a}{2} \in \mathfrak{A}.$$

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(c) $x\Box x$ has non-negative spectrum, and $\exp(it(x\Box x))$ is a 1-parameter group of isometries.

Finite dimensional Cartan factors

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- V,VI two exceptional Cartan factors in dimensions 16 and 27, which are no JC*-triples.

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Denote by $\langle \cdot, \cdot \rangle_r$ the canonical form, acting on the right of E.Then

$$\{x, y, z\} := \frac{\langle x, y \rangle_\ell z + x \langle y, z \rangle_r}{2}$$

turns E into a JB*-triple.

Ternary Rings of Operators

Hilbert C*-modules E are precisely those JB*-triple which embed into a C*-algebra $\mathfrak{A}$, and satisfy the more restrictive condition

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Objects in a category of Hilbert C*-modules in which morphisms are required to respect the product $[\cdot, \cdot, \cdot]$ are usually called *Ternary Rings of Operators* (TROs)

K-theory

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As it will turn out shortly, this is good enough.

First Stop: K-theory for TROs

To each TRO X there are left and right C*-algebras, $\mathcal{L}(X)$ and, respectively, $\mathcal{R}(X)$, acting on X. This point is of course obvious, when for you, TROs are Hilbert C*-modules.

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These algebras are Morita equivalent and so we may define

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This yields K-theory for TROs with all the well-known properties.

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Results of this calculation are next.

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Exceptional factors *The null space.*

Defining K-theory for JB*-triples

Definition

Denote by Ψ the functor that provides each JB*-triple with its enveloping TRO. Define, for a JB*-triple Z and a JB*-morphism ϕ,

$$K_*^{\mathrm{JB}^*}(Z) = K_*^{\mathrm{TRO}}(\Psi(Z)),$$

as well as

$$K_*^{\mathrm{JB}^*}(\phi) = K_*^{\mathrm{TRO}}(\Psi(\phi))$$

Defining K-theory for JB*-triples

Definition

Denote by Ψ the functor that provides each JB*-triple with its enveloping TRO. Define, for a JB*-triple Z and a JB*-morphism ϕ,

$$K_*^{\mathrm{JB}^*}(Z) = K_*^{\mathrm{TRO}}(\Psi(Z)),$$

as well as

$$K_*^{\mathrm{JB}^*}(\phi) = K_*^{\mathrm{TRO}}(\Psi(\phi))$$

This functor has the usual properties that one would expect from it, except, of course, stability, which already had a bad start.

K-groups of the Cartan factors

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This is not good enough for a classification, and more work is ahead.

Classification

Classifying finite dimensional TROs

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This is similar for TROs. The finite dimensional ones are all of the form $M_{m_1,n_1}(\mathbb{C}) \oplus M_{m_2,n_2}(\mathbb{C}) \oplus \cdots \oplus M_{m_k,n_k}(\mathbb{C})$. And so:

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Example

For a finite dimensional TRO as above, K-groups, enhanced by the scales of their left and right C*-algebras, are

$$(\mathbb{Z}, \underline{m}_1, \underline{n}_1) \oplus \cdots \oplus (\mathbb{Z}, \underline{m}_k, \underline{n}_k), \quad \underline{m} = \{1, \ldots, m\}$$

These objects do comprise sufficient information as to be classifying.

Isn't this all really about pairs of C*-algebras?

Example

The TROs $T = M_{1,2}(\mathbb{C}) \oplus M_{2,1}(\mathbb{C})$ and $U = M_{1,1}(\mathbb{C}) \oplus M_{2,2}(\mathbb{C})$ are non-isomorphic.

Nonetheless, we have $\mathcal{R}(T) = M_2 \oplus M_1 \simeq M_1 \oplus M_2 = \mathcal{R}(U)$, and $\mathcal{L}(T) = M_1 \oplus M_2 = \mathcal{L}(U)$
This yields the two non-isomorphic double-scaled ordered groups

$$K_0(T) = (\mathbb{Z}^2, \mathbb{N}_0^2, \{(0,0),(0,1),(0,2),(1,1),(1,2)\}, \{(0,0),(1,0),(1,1),(2,0),(2,1)\})$$

and

$$K_0(U) = (\mathbb{Z}^2, \mathbb{N}_0^2, \{(0,0),(0,1),(0,2),(1,1),(1,2)\}, \{(0,0),(0,1),(0,2),(1,1),(1,2)\}).$$

Tripotents and roots

Definition

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When Z is embedded into its enveloping TRO, a tripotent u becomes tripotent w.r.t. the TRO-product. But this means that the element $\langle u, u \rangle_\ell$ is a projection in the left C*-algebra of $TRO(Z)$ and thus gives rise to an element of $K_0^{\mathrm{JB}^*}(Z)$. We denote the the set of these classes in $K_0^{\mathrm{JB}^*}(Z)$ by Σ_Z.

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The root system of one of the Lie algebras underlying a (finite dimensional) JB*-triple, is very closely related to a certain subset ('grids') of the tripotent elements.

The JB*-scale we are discussing then is an imprint which root lattices leave on $K_0^{\mathrm{JB}*}(Z)$.

Ordering the K_0-group

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$(K_0(Z), K_0(Z)^+)$ then is an ordered group.

The K-JB* invariant

We finally come to the invariant that classifies the hermitian symmetric spaces in finite dimensions.

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Definition

The K-JB* invariant of a JB*-triple Z is the tuple

$$(K_0(Z), K_0(Z)^+, \Sigma_\ell, \Sigma_r, \Sigma_Z)$$

featuring both scales, Σ_ℓ, and Σ_r, of the enveloping $\mathrm{TRO}(Z)$ as well as the set Σ_Z of K-classes belonging to the tripotent elements $u \in Z$.

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The latter means that any morphism between K-JB*-invariants is induced from a JB*-morphism.

Note that no similar result holds for Dynkin diagrams.

The finite dimensional case

Theorem

The K-JB invariants for the finite dimensional JC*-factors are*

- $(\mathbb{Z}^2, \mathbb{N}_0^2, \underline{m} \oplus \underline{n}, \underline{n} \oplus \underline{m}, \underline{\min}\{m, n\})$, *rectangular $m \times n$-matrices*

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- $\left(\mathbb{Z}^N, \mathbb{N}_0^N, \bigoplus_{k=1}^N \underline{\binom{N}{k}}, \bigoplus_{k=1}^N \underline{\binom{N}{k-1}}, \binom{N-1}{0} \oplus \ldots \oplus \binom{N-1}{N-1}\right)$, *Hilbert space of dimension N*

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- $(\mathbb{Z}, \mathbb{N}_0, \underline{N}, \underline{N}, \underline{N})$, *symmetric factor in* M_N

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- $(\mathbb{Z}, \mathbb{N}_0, \underline{N}, \underline{N}, \underline{N})$, *symmetric factor in M_N*
- $(\mathbb{Z}, \mathbb{N}_0, \underline{N}, \underline{N}, \{2, 4, \cdots, 2k\})$, $2k \leq N < 2k+2$, *symplectic factor in M_N*

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- $\left(\mathbb{Z}, \mathbb{N}_0, \underline{2}^k, \underline{2}^k, \left\{2^{k-1}, 2^k\right\}\right)$, *$(2k+1)$-dimensional spin factor,*
  $\left(\mathbb{Z}^2, \mathbb{N}_0^2, \underline{2}^{k-1} \oplus \underline{2}^{k-1}, \underline{2}^{k-1} \oplus \underline{2}^{k-1}, \left\{2^{k-2}, 2^{k-1}\right\} \oplus \left\{2^{k-2}, 2^{k-1}\right\}\right)$,
  for the $2k$-dimensional spin factor.

More on inductive limits

Theorem

Each inductive limit of finite dimensional JB-triples Z admits a decomposition*

$$Z = H \oplus S \oplus P \oplus E$$

where H is a direct sum of Hilbert spaces, S a direct sum of spin factors and E a direct sum of exceptional factors.

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This classifies certain Kac-Moody algebras at the same time.