K-theory and Classification of Symmetric Spaces

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Adelaide, 22 March 2012

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All new results in the following, unaccounted for, are joint with Dennis Bohle.

Symmetric Spaces

Standard definition

Definitions of 'Symmetric Space' vary. The probably most traditional one:

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A Riemannian manifold M is called a *Riemannian symmetric space* if for each point $x \in M$ there exists an involution s_x (a 'symmetry') which is an isometry of M, and has x as an isolated fixed point.

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By elementary geometry, Euclidian space, the hyperbolic plane, or the (real) spheres are examples in this class of spaces. A beautiful result going back to E. Cartan states

Theorem

A Riemannian manifold M is locally symmetric iff its curvature tensor R is parallel, $\nabla R = 0$.

By appropriately composing symmetries, one can reach each point $m \in M$ from a given point $o \in M$ by an isometry. Symmetric spaces are hence homogeneous, and thus of the form G/K, where G is a Lie group and K is the isotropy group of a fixed point o.

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• a Lie group *G*, and

• an order 2 automorphism Σ of G (the master symmetry) Then G/K, where K denotes the fixed point group of Σ is a (Riemannian, if K is compact) symmetric space.

Lie algebra level

Denote by $\mathfrak g$ the Lie algebra of G, and by σ the derivative of Σ at the unit element of G. Then

$$\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{k}$$

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where $\mathfrak k$ and $\mathfrak m$ are the 1 and -1 eigenspaces of $\sigma,$ respectively.

This is the *Cartan decomposition* of \mathfrak{g} .

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The space \mathfrak{m} can be identified with the tangent space of G/K at the point eK. It is not a Lie subalgebra.

It has, however, the remarkable property that for any choice of elements $m_{1,2,3}\in\mathfrak{m}$,

 $[m_1, [m_2, m_3]] \in \mathfrak{m}$

For this reason, \mathfrak{m} is called a *Lie triple system*.

The hermitian case

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(I.e. a Kähler manifold which, as a Riemannian manifold, is symmetric)

A rough classification

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Both these halves can be divided further into symmetric spaces of compact and non-compact type, so that about 25% of the symmetric spaces are Hermitian and of non-compact type.

This latter class can be embedded as bounded symmetric domains into \mathbb{C}^n , for which the role of the isometry group is taken by the group of biholomorphic automorphisms.

Connection with C*-Algebras

An example

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The action of the group SU(n, n) on D_n is

$$Z \longmapsto (AZ + B)(A'Z + B')^{-1},$$

where the typical element $U \in SU(n, n)$ was written as

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These are the biholomorphic automorphisms and look quite like Möbius transforms.

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Then D can be written as

$$D = \operatorname{Aut}(D) / \operatorname{Iso}(\mathfrak{A})$$

where $Iso(\mathfrak{A})$ denotes the group of (Banach space) isometries of \mathfrak{A} . It coincides with the fixed points of an Aut(D), order 2 autmorphism Σ , defined through

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Thus, each such D is a symmetric space in a more general sense.

Approaching the C*-product

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Is there any relation between Lie triple and C*-products, both defined on $\mathfrak{m}?$

The answer to this question is yes.

Classification

JB*-Triple product

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$$[a, [b, c]] - i[a, [ib, c]] = \frac{ab^*c + cb^*a}{2}$$

Definition

A closed subspace of a C*-algebra \mathfrak{A} is called a *JC*-triple* iff for all $a, b, c \in \mathfrak{A}$ $\{a, b, c\} := \frac{ab^*c + cb^*a}{2} \in \mathfrak{A}.$

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(c) $x \Box x$ has non-negative spectrum, and $\exp(it(x \Box x))$ is a 1-parameter group of isometries.

Finite dimensional Cartan factors

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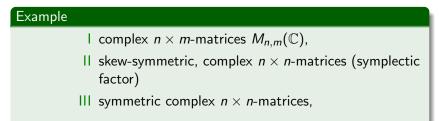
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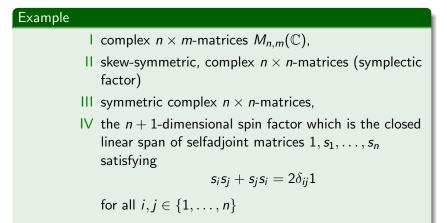
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- I complex $n \times m$ -matrices $M_{n,m}(\mathbb{C})$,
- II skew-symmetric, complex $n \times n$ -matrices (symplectic factor)

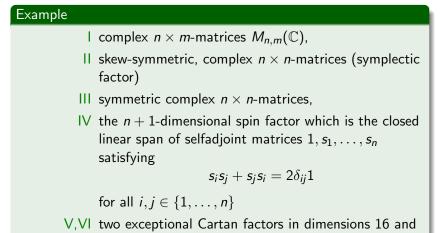
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27, which are no JC*-triples.

Hilbert C*-Modules

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- Let $(E, \langle \cdot, \cdot \rangle_{\ell})$ be a left (full) Hilbert C*-Module.
- (*Roughly:* A Hilbert space for which scalars come from a C^* -algebra)

Denote by $\langle\cdot,\cdot\rangle_r$ the canonical form, acting on the right of ${\it E}. Then$

$$\{x, y, z\} := \frac{\langle x, y \rangle_{\ell} z + x \langle y, z \rangle_{r}}{2}$$

turns *E* into a JB*-triple.

Ternary Rings of Operators

Hilbert C*-modules *E* are precisely those JB*-triple which embed into a C*-algebra \mathfrak{A} , and satisfy the more restrictive condition

$$[x, y, z] := xy^*z \in E$$
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Definition

Objects in a category of Hilbert C*-modules in which morphisms are required to respect the product $[\cdot, \cdot, \cdot]$ are usually called *Ternary Rings of Operators* (TROs)

K-theory

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As it will turn out shortly, this is good enough.

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These algebras are Morita equivalent and so we may define

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This yields K-theory for TROs with all the well-known properties.

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Results of this calculation are next.

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The enveloping TROs of the Cartan factors are the following $M_{m,n}(\mathbb{C}), m, n \geq 2$ $M_{m,n}(\mathbb{C}) \oplus M_{n,m}(\mathbb{C}),$ embedding $A \mapsto A \oplus A^{\top}$.

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Exceptional factors The null space.

Defining K-theory for JB*-triples

Definition

Denote by Ψ the functor that provides each JB*-triple with its enveloping TRO. Define, for a JB*-triple Z and a JB*-morphism ϕ ,

$$\mathcal{K}^{\mathrm{JB}^*}_*(Z) = \mathcal{K}^{\mathrm{TRO}}_*(\Psi(Z)),$$

as well as

$$\mathcal{K}^{\mathrm{JB}^*}_*(\phi) = \mathcal{K}^{\mathrm{TRO}}_*(\Psi(\phi))$$

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This functor has the usual properties that one would expect from it, except, of course, stability, which already had a bad start.

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- \mathbb{Z} (for both, symmetric and symplectic factors)
- Z (odd dimensional spin factors), and
 Z² (for the even-dimensional spin factors)

K-Theory

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- \mathbb{Z}^N (for Hilbert spaces of dimension N)
- \mathbb{Z} (for both, symmetric and symplectic factors)
- \mathbb{Z} (odd dimensional spin factors), and \mathbb{Z}^2 (for the even-dimensional spin factors)

This is not good enough for a classification, and more work is ahead.

Classification

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This is similar for TROs. The finite dimensional ones are all of the form $M_{m_1,n_1}(\mathbb{C}) \oplus M_{m_2,n_2}(\mathbb{C}) \oplus \cdots \oplus M_{m_k,n_k}(\mathbb{C})$. And so:

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Example

For a finite dimensional TRO as above, K-groups, enhanced by the scales of their left and right C*-algebras, are

$$(\mathbb{Z}, \underline{m}_1, \underline{n}_1) \oplus \cdots \oplus (\mathbb{Z}, \underline{m}_k, \underline{n}_k), \quad \underline{m} = \{1, \dots, m\}$$

These objects do comprise sufficient information as to be classifying.

Isn't this all really about pairs of C*-algebras?

Example

The TROs $T = M_{1,2}(\mathbb{C}) \oplus M_{2,1}(\mathbb{C})$ and $U = M_{1,1}(\mathbb{C}) \oplus M_{2,2}(\mathbb{C})$ are non-isomorphic.

Nonetheless, we have $\mathcal{R}(\mathcal{T}) = M_2 \oplus M_1 \simeq M_1 \oplus M_2 = \mathcal{R}(U)$, and $\mathcal{L}(\mathcal{T}) = M_1 \oplus M_2 = \mathcal{L}(U)$

This yields the two non-isomorphic double-scaled ordered groups

$$\begin{split} \mathcal{K}_0(\mathcal{T}) &= (\mathbb{Z}^2, \mathbb{N}^2_0, \{(0,0), (0,1), (0,2), (1,1), (1,2)\}, \\ &\{(0,0), (1,0), (1,1), (2,0), (2,1)\}) \end{split}$$

and

$$\begin{split} \mathcal{K}_0(U) &= (\mathbb{Z}^2, \mathbb{N}^2_0, \{(0,0), (0,1), (0,2), (1,1), (1,2)\},\\ &\{(0,0), (0,1), (0,2), (1,1), (1,2)\}). \end{split}$$

Definition

Let Z be a JB*-triple. An element $u \in Z$ is called *tripotent* iff $\{u, u, u\} = u$.

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When Z is embedded into its enveloping TRO, a tripotent u becomes tripotent w.r.t. the TRO-product. But this means that the element $\langle u, u \rangle_{\ell}$ is a projection in the left C*-algebra of TRO(Z) and thus gives rise to an element of $K_0^{\text{JB}^*}(Z)$. We denote the the set of these classes in $K_0^{\text{JB}^*}(Z)$ by Σ_Z .

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The JB*-scale we are discussing then is an imprint which root lattices leave on $K_0^{\text{JB*}}(Z)$.

Ordering the K_0 -group

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In order to treat morphisms, we still need the semigroup $K_0(Z)^+$, consisting of all classes of projections themselves, that is, the set obtained before, in the final step, the Grothendieck construction is applied in order to produce $K_0(Z)$.

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 $(K_0(Z), K_0(Z)^+)$ then is an ordered group.

The K-JB* invariant

We finally come to the invariant that classifies the hermitian symmetric spaces in finite dimensions.

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Definition

The K-JB* invariant of a JB*-triple Z is the tuple

$$(K_0(Z), K_0(Z)^+, \Sigma_\ell, \Sigma_r, \Sigma_Z)$$

featuring both scales, Σ_{ℓ} , and Σ_r , of the enveloping TRO(Z) as well as the set Σ_Z of K-classes belonging to the tripotent elements $u \in Z$.

Theorem

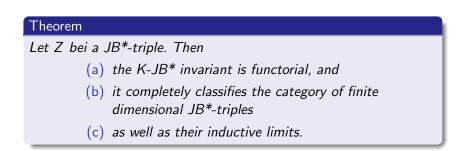
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Theorem

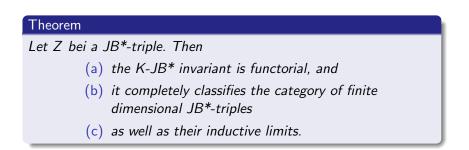
Let Z bei a JB*-triple. Then

- (a) the K-JB* invariant is functorial, and
- (b) it completely classifies the category of finite dimensional JB*-triples

(c) as well as their inductive limits.



The latter means that any morphism between K-JB*-invariants is induced from a JB*-morphism.



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Note that no similar result holds for Dynkin diagrams.

Theorem

The K-JB* invariants for the finite dimensional JC*-factors are

• $(\mathbb{Z}^2, \mathbb{N}^2_0, \underline{m} \oplus \underline{n}, \underline{n} \oplus \underline{m}, \underline{\min}\{m, n\})$, rectangular $m \times n$ -matrices

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- $\left(\mathbb{Z}^{N}, \mathbb{N}_{0}^{N}, \bigoplus_{k=1}^{N} {N \choose k}, \bigoplus_{k=1}^{N} {N \choose k-1}, {N-1 \choose 0} \oplus \ldots \oplus {N-1 \choose N-1}\right)$, Hilbert space of dimension N

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- $(\mathbb{Z}, \mathbb{N}_0, \underline{N}, \underline{N}, \{2, 4, \cdots, 2k\}), 2k \leq N < 2k + 2,$ symplectic factor in M_N

K-Theory

The finite dimensional case

Theorem

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- $(\mathbb{Z}, \mathbb{N}_0, \underline{N}, \underline{N}, \underline{N})$, symmetric factor in M_N
- (ℤ, ℕ₀, <u>N</u>, <u>N</u>, {2, 4, · · · , 2k}), 2k ≤ N < 2k + 2, symplectic factor in M_N

•
$$\left(\mathbb{Z}, \mathbb{N}_{0}, \underline{2}^{k}, \underline{2}^{k}, \left\{2^{k-1}, 2^{k}\right\}\right)$$
, $(2k+1)$ -dimensional spin factor,
 $\left(\mathbb{Z}^{2}, \mathbb{N}_{0}^{2}, \underline{2}^{k-1} \oplus \underline{2}^{k-1}, \underline{2}^{k-1} \oplus \underline{2}^{k-1}, \\ \left\{2^{k-2}, 2^{k-1}\right\} \oplus \left\{2^{k-2}, 2^{k-1}\right\}\right)$,

for the 2k-dimensional spin factor.

More on inductive limits

Theorem

Each inductive limit of finite dimensional JB*-triples Z admits a decomposition

 $Z=H\oplus S\oplus P\oplus E$

where H is a direct sum of Hilbert spaces, S a direct sum of spin factors and E a direct sum of exceptional factors.

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This classifies certain Kac-Moody algebras at the same time.