T-duality, Courant algebroids, and the exotic.

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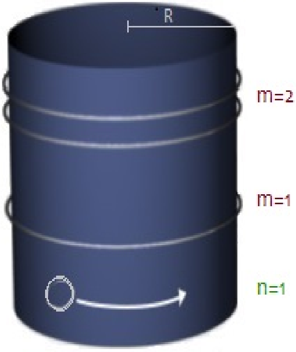
SUB CRUCE LUMEN

Overview

1. Topological T-duality in a H-flux
   - Isomorphism between the invariant, twisted differential forms on the dual spaces
2. Courant Algebroids and T-duality
   - Isomorphism between the invariant Courant algebroids on the dual spaces
3. Extension?
4. Exotic Differential Forms
5. Exotic Courant Algebroids

T-duality: a duality of string theory.

Consider the two dimensional space represented by a cylinder of radius R. Along the compact dimension, a string can have momentum, with momentum mode $n \in \mathbb{Z}$, but can also have a winding mode, $m \in \mathbb{Z}$.



$$M^2 = \left(\frac{n}{R}\right)^2 + \left(\frac{mR}{\alpha'}\right)^2 + \ldots$$

T-duality- a duality from string theory.

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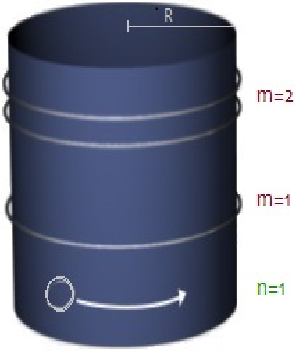

T-duality

$\longleftrightarrow$

$$m \leftrightarrow n$$

$$n \leftrightarrow m$$

$$R \leftrightarrow \frac{\alpha'}{R}$$

$$M^2 = \left(\frac{n}{R}\right)^2 + \left(\frac{mR}{\alpha'}\right)^2 + \ldots$$

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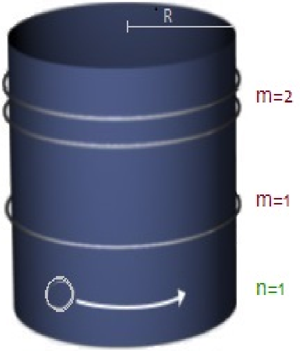

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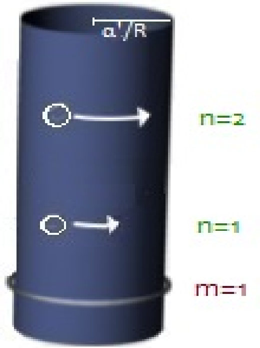

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This $R \to \frac{\alpha'}{R}$ duality of toroidal compactification extends to more complicated, curved spacetimes which possess an abelian group of isometries, where the metric and relevant fields are transformed via the Buscher rules which are locally defined.

Given such rules are only defined locally, the question then is how does T-duality effect the global data such as the topology of the space?

For what follows we are now going to restrict our attention to manifolds with a single compact dimension which take the structure of principal circle bundle.

T-duality for the principal circle bundle.

We begin with a principal circle bundle,

$$\begin{array}{ccc} S^1 & \hookrightarrow & Z \\ & & \downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

whose first Chern class is $[F] = c_1(Z) \in H^2(M, \mathbb{Z})$, along with a background H-flux $[H] \in H^3(Z, \mathbb{Z})$.

The H-flux arises from string theory as the curvature of the B-field, one of the massless fields, alongside the dilaton and the famous graviton. The R-R fields (the objects to which D-branes couple) will also be significant in our analysis of T-duality. These are given by the even/odd classes of the H-twisted cohomology classes over Z.

T-duality for the principal circle bundle.

Original data.

$$\begin{tikzcd} S^1 \arrow[r, hook] & Z \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

$([F], [H]) \in H^2(M, \mathbb{Z}) \times H^3(Z, \mathbb{Z}).$

T-dual data.

$$\begin{tikzcd} \hat{S}^1 \arrow[r, hook] & \hat{Z} \arrow[d, "\hat{\pi}"] \\ & M \end{tikzcd}$$

$([\hat{F}], [\hat{H}]) \in H^2(M, \mathbb{Z}) \times H^3(\hat{Z}, \mathbb{Z}).$

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T-dual data.

$$\begin{array}{ccc} \hat{S}^1 & \hookrightarrow & \hat{Z} \\ & & \downarrow{\scriptstyle \hat{\pi}} \\ & & M \end{array}$$

$([\hat{F}],[\hat{H}]) \in H^2(M,\mathbb{Z})\times H^3(\hat{Z},\mathbb{Z})$.

T-duality for the principal circle bundle. (Bouwknegt, Evslin, Mathai 2004).

Here the bundle $\hat{Z}$ is defined such that $c_1(\hat{Z}) := \pi_*([H])$, where π_* denotes the pushforward map. The dual data is then defined by the relations:

$$[\hat{F}] = \pi_*([H]), \tag{1}$$

$$[F] = \hat{\pi}_*([\hat{H}]). \tag{2}$$

Let $Z \times_M \hat{Z} = \{(a,b) \in Z \times \hat{Z} | \pi(a) = \hat{\pi}(b)\}$, then given the maps

$$\begin{tikzcd} & Z \times_M \hat{Z} \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & \\ Z \arrow[dr, "\pi"'] & & \hat{Z}, \arrow[dl, "\hat{\pi}"] \\ & M & \end{tikzcd}$$

if the two H-fluxes, $[H]$ and $[\hat{H}]$, satisfy

$$p^*([H]) = \hat{p}^*([\hat{H}]),$$

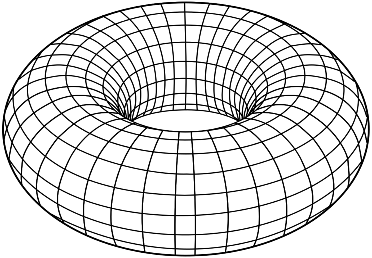
then the dual pair is unique up to bundle automorphism.

Example.

Original data.

$$\begin{tikzcd} S^1 \arrow[r, hook] & S^2 \times S^1 \arrow[d, "\pi"] \\ & S^2 \end{tikzcd}$$

$$([0],[1]) \in H^2(M,\mathbb{Z}) \times H^3(Z,\mathbb{Z}).$$



$+1$

T-dual data.

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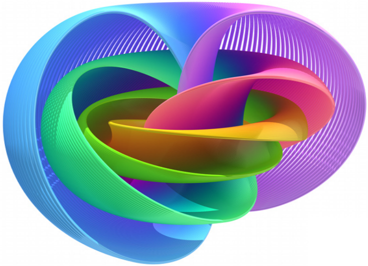

T-duality of H-twisted cohomology.

Theorem (Bouwknegt-Evslin-Mathai,(2004))

Let A, $\hat{A}$ denote connection forms on Z and $\hat{Z}$ respectively, choose some invariant representative $H \in [H]$ and $\hat{H} \in [\hat{H}]$, and let $(\Omega^\bullet(Z)^{S^1}, d + H)$[1] *denote the H-twisted $\mathbb{Z}_2$-graded differential complex.*
Then the following map:

$$\begin{aligned} \tau : (\Omega^\bullet(Z)^{S^1}, d + H) &\to (\Omega^{\bullet+1}(\hat{Z})^{\hat{S}^1}, d + \hat{H}) \\ \omega &\mapsto \int_{S^1} e^{A\wedge\hat{A}} \wedge \omega, \end{aligned}$$

is a chain map isomorphism between the twisted, $\mathbb{Z}_2$-graded complexes. Furthermore, this induces an isomorphism on the twisted cohomology:

$$\tau : H^\bullet_{d+H}(Z) \to H^{\bullet+1}_{d+\hat{H}}(\hat{Z}).$$

[1] Note: $\Omega^*(Z)^{S^1} = \{\omega \in \Omega^*(Z) | \mathcal{L}_v(\omega) = 0\}$, where v is an invariant period-1 generator of the circle action, and $\mathcal{L}_v$ denotes the Lie derivative along v.

Courant algebroids

A *Courant algebroid* is given by a vector bundle $E \to M$ equipped with a nondegenerate, symmetric bilinear form

$$\langle \cdot, \cdot \rangle : \Gamma(E) \otimes \Gamma(E) \to C^\infty(M),$$

a bilinear (Dorfman) bracket

$$[\cdot, \cdot] : \Gamma(E) \otimes \Gamma(E) \to \Gamma(E),$$

and a smooth bundle map $\rho : E \to TM$ called the *anchor*, satisfying:

1. $[a, [b, c]] = [[a, b], c] + [b, [a, c]]$,
2. $\rho([a, b]) = [\rho(a), \rho(b)]_{\mathrm{LB}}$,
3. $[a, fb] = \rho(a)(f)b + f[a, b]$,
4. $[a, b] + [b, a] = d\langle a, b\rangle$, where d denotes the differential,
5. $\rho(a)\langle b, c\rangle = \langle [a, b], c\rangle + \langle b, [a, c]\rangle$,

where $a, b, c \in \Gamma(E)$, $f \in C^\infty(M)$, and $d : C^\infty(M) \to \Gamma(E)$ is the induced differential operator defined by

$$\langle df, a\rangle = \rho(a)f.$$

Example 1

Let Z denote a manifold, let $H \in \Omega^3_{\mathrm{cl}}(Z)$, and consider the vector bundle,

$$\pi : TZ \oplus T^*Z \to Z.$$

Given sections $X + \alpha, Y + \beta \in \Gamma(TZ \oplus T^*Z)$, this bundle along with the following data defines a Courant algebroid:

- *Bracket* : $[X + \alpha, Y + \beta]_H = [X, Y]_{\mathrm{LB}} + \mathcal{L}_X \beta - \iota_Y d\alpha + \iota_X \iota_Y H$
- *Inner product* : $\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\alpha(Y) + \beta(X))$
- *Anchor map* : $\rho(X + \alpha) = X$
- *Induced differential operator* : $d : C^\infty(M) \to \Gamma(TZ \oplus T^*Z)$ (exterior derivative)

- *Clifford module on $\Omega(Z)$* : $(X + \alpha) \cdot \omega = \iota_X \omega + \alpha \wedge \omega$

• The bracket defined above is the derived bracket with respect to the twisted differential operator $d + H$. That is,

$$[X + \alpha, Y + \beta]_H := [[d + H, X + \alpha], Y + \beta].$$

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Example 2

Let $\pi : Z \to M$ to be a principal circle bundle, with invariant H-flux representative H. Now consider only the invariant sections of the Courant algebroid $\bar{\pi} : TZ \oplus T^*Z \to Z$, i.e., the sections $X + \alpha \in \Gamma(Z, TZ \oplus T^*Z)$ satisfying $\mathcal{L}_v(X + \alpha) = 0$. These sections are closed under our previously defined bracket $[\, ,\,]_H$.

Any invariant section $X + \alpha \in \Gamma(Z, TZ \oplus T^*Z)^{S^1}$ may be expressed as

$$X + \alpha = h_A(x) + fv + \pi^*(a) + gA, \tag{3}$$

for some $x \in \Gamma(M, TM)$, $a \in \Omega^1(M)$, $f, g \in C^\infty(M)$, or equivalently:

$$(x, f, a, g) \in \Gamma(M, TM \oplus \mathbb{1}_{\mathbb{R}} \oplus T^*M \oplus \mathbb{1}_{\mathbb{R}}). \tag{4}$$

(We will often write this section, conveniently, yet incorrectly, as $x + fv + a + gA$).

As a result, we get an invariant Courant algebroid, $((TZ \oplus T^*Z)^{S^1}, [\cdot, \cdot]_H)$, of the bundle

$$\tilde{\pi} : TM \oplus \mathbb{1}_{\mathbb{R}} \oplus T^*M \oplus \mathbb{1}_{\mathbb{R}} \to M.$$

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T-duality of Courant algebroids

Theorem (Cavalcanti-Gaultieri,(2010))

The following map:

$$\phi : ((TZ \oplus T^*Z)^{S^1}, [\cdot,\cdot]_H) \to ((T\hat{Z} \oplus T^*\hat{Z})^{S^1}, [\cdot,\cdot]_{\hat{H}})$$
$$(x, f, a, g) \mapsto (x, g, a, f),$$

defines an isomorphism between Courant algebroids.
In addition, given the T-duality isomorphism τ between invariant differential forms, we get that τ defines an isomorphism of Clifford modules, so:

$$\tau(x \cdot \omega) = \phi(x) \cdot \tau(\omega),$$

where $x \in \Gamma(TZ \oplus T^*Z)^{S^1}$ and $\omega \in \Omega^\bullet(Z)^{S^1}$.

T-duality and Invariance

Can we extend these isomorphisms τ and ϕ to include not only the invariant data, but the non-invariant as well. Do we get a T-duality map arising as an extension of the invariant one?

$$\bar{\tau} : (\Omega^*(Z), d + H) \xrightarrow{\text{T-duality}} \quad \textbf{?}$$

$$\bar{\phi} : (TZ \oplus T^*Z, [\cdot, \cdot]_H) \xrightarrow{\text{T-duality}} \quad \textbf{?}$$

What are these objects on the right hand side? Exotic!

Differential forms: The local picture

Key fact we are going to use is that given any differential form $\omega \in \Omega^*(Z)$, we may apply a family Fourier expansion to transform it into the form

$$\omega = \sum_{n \in \mathbb{Z}} \omega_n,$$

where $\omega_n \in \Omega^*_n(Z) := \{\alpha \in \Omega^*(Z)) | \mathcal{L}_v(\alpha) = n\alpha\}$, where v denotes the invariant period-1 generator of the circle action

Now let $\{U_\alpha\}$ be a good cover of M, such that $\pi^{-1}(U_\alpha) \cong U_\alpha \times S^1$. Then we can express the form ω_n locally by:

$$\omega_n|_{U_\alpha} = (\omega_{\alpha,n,1} + \omega_{\alpha,n,0} \wedge A) e^{2\pi i n \theta_\alpha},$$

where A is the connection on Z, and $\omega_{\alpha,n,1}$, $\omega_{\alpha,n,0}$ define basic forms on $\pi^{-1}(U_\alpha)$.

Twisted integration

If we simply use the previous formula of T-duality we had for the invariant differential forms,

$$\omega_n \quad \mapsto \int_{S^1} e^{A \wedge \hat{A}} \wedge \omega_n = 0,$$

we get that the dual is trivial when $n \neq 0$.
The key is to include a *twisting* to the integral, to take into account the lack of invariance. That is, something locally of the form

$$\int_{S^1} \left(e^{A \wedge \hat{A}} \wedge \omega_n\right) e^{-2\pi i n \theta_\alpha}.$$

But this isn't globally well defined. To achieve this, we they do is tensor this form with a certain section of a line bundle, resulting in a well-defined invariant, line bundle-valued differential form.

Exotic differential forms (Han, Mathai 2018)

Let $p : \xi \to M$ denote the associated line bundle of Z, with the induced connection ∇^{ξ} induced from A. Let $\{s_\alpha\}$ denote a local basis of ξ over $\{U_\alpha\}$ corresponding to the constant map $U_\alpha \to \{1\} \subset S^1$.

The *exotic differential forms*, denoted $\mathcal{A}^\bullet(\hat{Z})^{\hat{S}^1}$, are defined as a collection of invariant, line bundle valued differential forms, given by:

$$\mathcal{A}^\bullet(\hat{Z})^{\hat{S}^1} = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}^\bullet_n(\hat{Z})^{\hat{S}^1} := \bigoplus_{n\in\mathbb{Z}} \Omega^\bullet(\hat{Z}, \hat{\pi}^*(\xi^{\otimes n}))^{\hat{S}^1}.$$

Along with the differential operator defined on the n-th weight space by

$$-(\hat{\pi}^*\nabla^{\xi^{\otimes n}} - \iota_{n\hat{v}} + \hat{H}) : \mathcal{A}^\bullet_n(\hat{Z})^{\hat{S}^1} \to \mathcal{A}^{\bullet+1}_n(\hat{Z})^{\hat{S}^1},$$

we get a $\mathbb{Z}_2$-graded complex $(\mathcal{A}^\bullet_n(\hat{Z})^{\hat{S}^1}, -(\hat{\pi}^*\nabla^{\xi^{\otimes n}} - \iota_{n\hat{v}} + \hat{H}))$.

- Note that the space of sections of the invariant Courant algebroid wrt $\hat{Z}$ and $\hat{H}$, defines a Clifford action on the exotic differential forms.

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• Note that the space of sections of the invariant Courant algebroid wrt $\hat{Z}$ and $\hat{H}$, defines a Clifford action on the exotic differential forms.

T-duality of twisted differential forms

Theorem (Han-Mathai, (2018))

The following map $\bar{\tau} = \oplus_n \bar{\tau}_n : \Omega^\bullet(Z) \to \mathcal{A}^{\bullet+1}(\hat{Z})^{\hat{S}^1}$, *where*

$$\bar{\tau}_n = \Omega^\bullet_{-n}(Z) \to \mathcal{A}_n^{\bullet+1}(\hat{Z})^{\hat{S}^1},$$

which can be defined locally by

$$\bar{\tau}_n(\omega_{-n})|_{U_\alpha} = \left(\int^{\pi^{-1}(U_\alpha)/U_\alpha} (e^{A\wedge\hat{A}} \wedge \omega_{-n,\alpha}) e^{2\pi i n \theta_\alpha} \right) \otimes \pi^*(s_\alpha)^{\otimes n},$$

is an isomorphism of Clifford algebras which extends τ, *so for any* $X + \alpha \in (TZ \oplus T^*Z)^{S^1}$

$$\bar{\tau}((X+\alpha)\cdot\omega) = \phi(X+\alpha)\cdot\bar{\tau}(\omega).$$

Furthermore, it induces a chain map on the weighted complexes:

$$\bar{\tau}_n : (\Omega^\bullet_{-n}(Z), d + H) \to (\mathcal{A}_n^{\bullet+1}(\hat{Z})^{S^1}, -(\hat{\pi}^*\nabla^{\xi^{\otimes n}} - i_{n\hat{v}} + \hat{H})).$$

Extending Courant algebroids

We want to construct a map

$$\bar{\phi} : (TZ \oplus T^*Z, [\cdot,\cdot]_H) \xrightarrow{\text{T-duality}} \quad \textbf{?}$$

which extends the invariant Courant algebroid isomorphism

$$\phi : ((TZ \oplus T^*Z)^{S^1}, [\cdot,\cdot]_H) \to ((T\hat{Z} \oplus T^*\hat{Z})^{S^1}, [\cdot,\cdot]_{\hat{H}})$$

in such a way that the exotic Courant algebroid defines an action on our exotic differential forms, so that for any $X + \alpha \in \Gamma(TZ \oplus T^*Z)$,

$$\bar{\tau}((X+\alpha)\cdot\omega) = \bar{\phi}(X+\alpha)\cdot\bar{\tau}(\omega).$$

Local picture

Any section $X+\alpha \in \Gamma(TZ \oplus T^*Z)$ may be Fourier expanded into the form

$$X+\alpha = \sum_{n\in\mathbb{Z}} (X+\alpha)_n$$

where $(X+\alpha)_n \in \Gamma_n(TZ \oplus T^*Z) := \{s \in \Gamma(TZ \oplus T^*Z) | \mathcal{L}_v(s) = ns\}$, which can then be expressed locally as:

$$(X+\alpha)_n|_{U_\alpha} = e^{2\pi i n\theta_\alpha}(x_{\alpha,n} + f_{\alpha,n}v + a_{\alpha,n} + g_{\alpha,n}A),$$

where $x_{\alpha,n}$ is a horizontal v.f, $a_{\alpha,n}$ is a basic form, $f_{\alpha,n}, g_{\alpha,n} \in C^\infty(U_\alpha)$.

Now consider the following invertible map:

$$\mu_n : \Gamma_{-n}(TZ \oplus T^*Z) \to \Gamma((TM \oplus \mathbb{1} \oplus T^*M \oplus \mathbb{1}) \otimes \xi^{\otimes n})$$

which can be defined locally as

$$\mu_n(e^{-2\pi i n\theta_\alpha}(X_\alpha + f_\alpha v + \alpha_\alpha + g_\alpha A)) = (X_\alpha, f_\alpha, \alpha_\alpha, g_\alpha) \otimes s_\alpha^{\otimes n}$$

$$\left(\cong (X_\alpha + f_\alpha v + \alpha_\alpha + g_\alpha A) \otimes \pi^*(s_\alpha^{\otimes n}) \right)$$

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Exotic Courant algebroid

Definition

Let M be a manifold, E be a Courant algebroid over M, L a line bundle over M, and let $\mathcal{L} = \oplus_n(L^{\otimes n})$.

An *exotic* Courant algebroid over M is given by an infinite-dimensional vector bundle

$$\mathcal{E} = \bigoplus_{n \in \mathbb{Z}} (E \otimes L^n) \to M,$$

a bilinear map $\langle \ , \ \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{L}$, a bilinear bracket $[\ ,\ ] : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$, an anchor map $\rho : \mathcal{E} \to TM \otimes \mathcal{L}$, and an induced differential operator $D : \Gamma(\mathcal{L}) \to \Gamma(\mathcal{E})$ defined by the relation

$$\langle Df, a \rangle = \rho(a)f,$$

such that for all $a, b, c \in \mathcal{E}$ and $f \in \Gamma(\mathcal{L})$, we get

- $[a, [b, c]] = [[a, b], c] + [b, [a, c]]$
- $[a, fb] = \rho(a)(f)b + f[a, b]$
- $\rho(a)\langle b, c \rangle = \langle [a, b], c \rangle + \langle b, [a, c] \rangle$.
- $\rho([a, b]) = [\rho(a), \rho(b)]_*$
- $[a, b] + [b, a] = D\langle a, b \rangle$

Example 1

Lets reconsider the infinite bundle :

$$p : \bigoplus_{n\in\mathbb{Z}} (TM \oplus \mathbb{1} \oplus T^*M \oplus \mathbb{1}) \otimes \xi^{\otimes n} \to M.$$

where we can equivalently view sections of this bundle as sections of the bundle $\oplus_n \Gamma((TZ \oplus T^*Z) \otimes \pi^*(\xi^{\otimes n}))^{S^1}$.

Now take the differential operator $D = \oplus D_n$, where

$$D_n := \nabla^{\otimes n} - nA : \Gamma(M, \xi^{\otimes n}) \to \Gamma(M, (TM \oplus \mathbb{1} \oplus T^*M \oplus \mathbb{1}) \otimes \xi^{\otimes n}),$$

and define the bracket to be the derived bracket of the operator $D + H$ acting on $\oplus_n \Omega^\bullet(M, \xi^{\otimes n})$. Take the inner product to be defined such that

$$\langle a \otimes s_1^{\otimes n}, b \otimes s_2^{\otimes m} \rangle = \langle a, b \rangle_{\mathrm{CA}}\, s_1^{\otimes n} \otimes s_2^{\otimes m},$$

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Now returning to the bijection we previously defined:

$$\mu : \Gamma(TZ \oplus T^*Z) \to \Gamma\big(\oplus_n (TM \oplus \mathbb{1} \oplus T^*M \oplus \mathbb{1}) \otimes \xi^{\otimes n}\big),$$

we find that

$$\begin{aligned} [\mu(a), \mu(b)]_{ECA} &= \mu([a, b]_{CA}), \\ \langle \mu(a), \mu(b) \rangle_{ECA} &= \varphi(\langle a, b \rangle_{CA}), \\ \rho(\mu(a))_{ECA} &= \rho(a)_{CA}, \end{aligned}$$

where we use $\varphi : C^\infty(Z) \xrightarrow{\cong} \oplus_n \Gamma(\xi^{\otimes n})$, so that locally $\varphi_\alpha(e^{-2\pi n \theta_\alpha}) = s^{\otimes n}$.

So we get a transfer of the Courant algebroid structure on $TZ \oplus T^*Z$ to an exotic Courant algebroid structure on
$(\oplus_n((TZ \oplus T^*Z) \otimes \pi^*(\xi^{\otimes n}))^{S^1}, [\cdot, \cdot]_H)$.

Extended T-duality

We want to construct a map

$$\bar{\phi} : (TZ \oplus T^*Z, [\cdot,\cdot]_H) \xrightarrow{\text{T-duality}} \quad \textbf{?}$$

which equivalently defines a map

$$\bar{\phi} : (\oplus_n((TZ \oplus T^*Z) \otimes \pi^*(\xi^{\otimes n}))^{S^1}, [\cdot,\cdot]_H) \xrightarrow{\text{T-duality}} \quad \textbf{?}$$

Example 2

Now we will consider the elements of $\oplus_n \Gamma((T\hat{Z} \oplus T^*\hat{Z}) \otimes \hat{\pi}^*(\xi^{\otimes n}))^{\hat{S}^1}$. Given we are only interested in the invariant sections, we can again view these as sections of the bundle,

$$p : \oplus_n((TM \oplus \mathbb{1} \oplus T^*M \oplus \mathbb{1}) \otimes \xi^{\otimes n}) \to M.$$

Now define the differential operator $\hat{D}$ to act on the weight spaces by

$$\hat{D}_n := \nabla^{\otimes n} - i_{n\hat{v}} : \Gamma(M, \xi^{\otimes n}) \to \Gamma(M, (TM \oplus \mathbb{1} \oplus T^*M \oplus \mathbb{1}) \otimes \xi^{\otimes n}).$$

Take the bracket to be the derived bracket of the operator $\hat{D} + \hat{H}$, denoted $[\cdot, \cdot]_{\hat{H}}$, and take the inner product such that

$$\langle a \otimes s_1^{\otimes n}, b \otimes s_2^{\otimes m} \rangle = \langle a, b \rangle_{CA} \, s_1^{\otimes n} \otimes s_2^{\otimes m}.$$

Lastly, take the anchor map to be defined by the relation

$$\langle \hat{D} f, a \rangle = \hat{\rho}(a) f,$$

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Exotic Courant algebroids and T-duality

Theorem (Joint work with Mathai)

The following map

$$\bar{\phi} : (TZ \oplus T^*Z, [\cdot,\cdot]_H) \xrightarrow{T\text{-duality}} (\oplus_n((T\hat{Z} \oplus T^*\hat{Z}) \otimes \hat{\pi}^*(\xi^{\otimes n}))^{S^1}, [\cdot,\cdot]_{\hat{H}})$$
$$(X, f, \alpha, g) \otimes s^n \mapsto (X, g, \alpha, f) \otimes s^n,$$

defines an isomorphism between exotic Courant algebroids. Furthermore, it extends the Courant algebroid isomorphism

$$\phi : ((TZ \oplus T^*Z)^{S^1}, [\cdot,\cdot]_H) \to ((T\hat{Z} \oplus T^*\hat{Z})^{S^1}, [\cdot,\cdot]_{\hat{H}}),$$

in such a way that the exotic Courant algebroid has a natural action on the exotic differential forms, such that for any $X + \alpha \in \Gamma(TZ \oplus T^*Z)$,

$$\bar{\tau}((X+\alpha)\cdot\omega) = \bar{\phi}(X+\alpha)\cdot\bar{\tau}(\omega).$$

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