A Rescaled Spinor Bundle on the Tangent Groupoid

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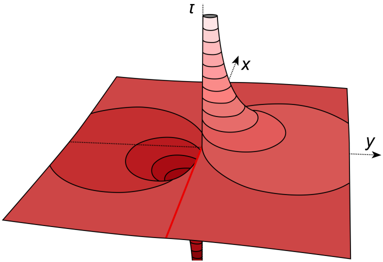
Introduction



I'm going to talk about a construction that brings together the local and the K-theory approaches to the proof of the index theorem. It uses the tangent groupoid which is a special case of the deformation to the normal cone construction.

w. Ahmad Reza Haj Saeedi Sadegh. *Euler-like vector fields, deformation spaces and manifolds with filtered structure.* Doc. Math. **23** (2018) 293-325.

w. Zelin Yi. *Spinors and the tangent groupoid.* Doc. Math., to appear. arXiv 1902.08351

The Symbol and the Index of an Elliptic Operator

Let D be an order p linear partial differential operator on a closed manifold M.

Let m be a point in M, and denote by D_m the constant coefficient operator obtained by freezing coefficients at m, and dropping lower order terms. The model operator D_m is well-defined as a differential operator on T_mM.

Using Fourier transform, D_m may be viewed as a homogeneous polynomial function on T^*_mM. And D is elliptic if (for every m) this symbol function is nowhere zero on $T_mM \setminus \{0\}$.

Gelfand's Problem: Find a formula for the Fredholm index of D in terms of the symbol of D.

This was solved by Atiyah and Singer using the formalism of topological K-theory.

Topological Index Map

Two key insights:

From the symbol one may construct a symbol class in topological K-theory.

$$\sigma(D) \in K(T^*M) \quad \text{or} \quad \sigma(D) \in K(C^*(TM))$$

(for the latter, think of TM as a bundle of Lie groups over M).

There is an analytic index map

$$K(C^*(TM)) \longrightarrow \mathbb{Z}$$

that maps $\sigma(D)$ to $\mathrm{Index}(D)$.

(*A third: Atiyah and Singer already "knew" the formula for this map.*)

Local Approach to the Index Theorem

For simplicity, assume that D is a first-order elliptic operator like the Dirac operator. Spectral theory shows that

$$\mathrm{Index}(D) = \mathrm{Tr}\big(\exp(-tD^*D)\big) - \mathrm{Tr}\big(\exp(-tDD^*)\big)$$

for any and all $t > 0$.

For a Laplace-type operator Δ on M^{2n} such as D^*D or DD^* it may be shown that

$$\mathrm{Tr}\big(\exp(-t\Delta)\big) = \int_M k_t(m, m)\, dm,$$

where the integral kernel has an asymptotic expansion

$$k_t(m, m) \sim a_{-n}(m)t^{-n} + a_{-(n-1)}(m)t^{-(n-1)} + \cdots$$

Local index strategy: Compute the terms $a_0(m)$ for $\Delta = D^*D$ and $\Delta = DD^*$; take the difference; and integrate over M.

This is easier said than done!

Getzler's Proof of the Index Theorem

This remarkable method works for the Dirac operator

$$\not{D} = \begin{bmatrix} 0 & D^* \\ D & 0 \end{bmatrix}$$

on a Riemannian spin manifold M^{2n}, defined using the Riemannian spin connection. It shows that in the local formula

$$\mathsf{STr}\big(\exp(-t\not{D}^2)\big) = \int_M \mathsf{str}(k_t(m,m))\, dm$$

there is an asymptotic expansion

$$\mathsf{str}(k_t(m,m)) \sim a_0(m)t^0 + a_1(m)t^1 + a_2(m)t^2 + \cdots$$

There are no singular terms. And it provides a direct formula for $a_0(m)$ in terms of the Riemann curvature tensor.

The Tangent Groupoid—A First Look

I'll describe the *families point of view* on the tangent groupoid, and on Lie groupoids generally.

Let M be a smooth manifold. The tangent groupoid is a certain smooth manifold $\mathbb{T}M$ that is equipped with a submersion

$$s\colon \mathbb{T}M \longrightarrow M \times \mathbb{R}$$

(this is the source fibration). The fibers are

$$\mathbb{T}M_{(m,t)} \cong \begin{cases} M & t \neq 0 \\ T_m M & t = 0. \end{cases}$$

The remaining structural features of $\mathbb{T}M$ make it possible to speak of an equivariant family of operators on the source fibers.

Differential Operators and the Tangent Groupoid

$$\mathbb{T}M_{(m,t)} \cong \begin{cases} M & t \neq 0 \\ T_m M & t = 0 \end{cases}$$

Theorem

If D is a differential operator on M of order q, then the operators

$$D_{(m,t)} = \begin{cases} t^q D & t \neq 0 \\ D_m & t = 0 \end{cases}$$

constitute, under the identifications above, a smooth and equivariant family of differential operators on the source fibers of the tangent groupoid.

Tangent Groupoid—Five Minute University Version

The tangent groupoid gives a geometric context in which an operator D and its symbol $\sigma(D)$ are combined into a single entity.

If D is elliptic and M is closed, then using techniques pioneered by Alain Connes, both the K-theoretic symbol class and the analytic index may be recovered from this entity, *using more or less the same mechanism*.

This doesn't by itself solve Gelfand's problem, but it goes a long way in that direction.

Deformation to the Normal Cone

Let M be a smooth, embedded submanifold of a smooth manifold V.

Form the algebra $A(\mathbb{N}_V M)$ of all Laurent polynomials

$$\sum a_p t^{-p}$$

where each a_p is a smooth function on V, and a_p vanishes to order $\geq p$ on M.

Define $\mathbb{N}_V M = \mathsf{CharSpec}(A(\mathbb{N}_V M))$.

Then

$$\mathbb{N}_V M = N_V M \times \{0\} \ \sqcup \ \bigsqcup_{t \neq 0} V \times \{t\}$$

and each $f \in A(\mathbb{N}_V M)$ is a “regular” function on $\mathbb{N}_V M$.

Functions in the Coordinate Algebra

$$\boxed{\mathbb{N}_V M = N_V M \times \{0\} \ \sqcup \ \bigsqcup_{t \neq 0} V \times \{t\}}$$

Smooth functions on V (times t^0) belong to $A(\mathbb{N}_V M)$:

$$\begin{cases} a\colon (v,t) \mapsto a(v) \\ a\colon (X_m, 0) \mapsto a(m) \end{cases}$$

Smooth functions on V that vanish on M, times t^{-1}, belong to $A(\mathbb{N}_V M)$:

$$\begin{cases} at^{-1}\colon (v,t) \mapsto a(v)/t \\ at^{-1}\colon (X_m, 0) \mapsto X_m(a) \end{cases}$$

Exponentials

Let $A_0(\mathbb{N}_V M)$ be the quotient of $A(\mathbb{N}_V M)$ by the ideal generated by t.

Let X be a vector field on V. The formula

$$\boldsymbol{X}\colon \sum X(a_p)t^{-p} \longmapsto \sum a_p t^{-(p-1)}$$

is a derivation, and

$$\exp(\boldsymbol{X})\colon A_0(\mathbb{N}_V M) \longrightarrow A_0(\mathbb{N}_V M)$$

is an automorphism. If $a \in A_0(\mathbb{N}_V M)$, then

$$a(X_m, 0) = \exp(\boldsymbol{X})(a)(0_m, 0).$$

Functoriality and the Tangent Groupoid

The tangent groupoid of M is the deformation space $\mathbb{N}_{M^2}M$ for the diagonal embedding of M in its square:

$$\mathbb{T}M = TM \times \{0\} \;\sqcup\; \bigsqcup_{t \neq 0} M \times M \times \{t\}.$$

The deformation space construction is a functor from submanifolds to manifolds over $\mathbb{R}$, so from

$$\begin{array}{ccc} M^2 & \rightrightarrows & M \\ \uparrow & & \uparrow \\ M & \xrightarrow[=]{} & M \end{array}$$

we obtain source and target maps

$$t, s \colon \mathbb{T}M \rightrightarrows M \times \mathbb{R}.$$

The remaining groupoid structure is obtained similiarly from the pair groupoid $M^2 \rightrightarrows M$.

Differential Operators, Again, and Order of Vanishing

Lemma

A smooth function $f \colon M \times M \to \mathbb{R}$ vanishes to order p on M if and only if Df vanishes on M for every differential operator D on M (acting of the first factor of $M \times M$) of order $(p-1)$ or less.

Theorem

Let M be a smooth manifold and let D be a linear partial differential operator on M of order q. The formula

$$D_{(m,\lambda)} = \begin{cases} t^q D & t \neq 0 \\ D_m & t = 0 \end{cases}$$

defines a smooth and equivariant family of differential operators on the source fibers of $\mathbb{T}M$.

Proof.

The action of $t^q D$ on the first factor preseves $A(\mathbb{T}M)$. □

Getzler Order of a Differential Operator

From now on M will be an even-dimensional spin manifold, with spinor bundle S and Riemannian spin connection ∇.

The following definition applies to any differential operator acting on sections of S.

Definition

A differential operator has Getzler order $\leq p$ if it can be expressed as a finite sum of operators of the form

$$f \cdot D_1 \cdots D_p,$$

where f is a smooth function, and each D_j is some ∇_X, or some $c(X)$, or the identity operator.

Towards a Rescaled Spinor Bundle

Our first aim is to construct a module over $A(\mathbb{T}M)$ in much the same way as $A(\mathbb{T}M)$ itself is constructed—using Laurent polynomials and a notion of order of vanishing on the diagonal in $M \times M$.

The following definition uses $S_m \otimes S_m^* \cong \mathrm{Cliff}(T_m M)$.

Definition

A smooth section of $S \boxtimes S^*$ has Clifford order $\leq d$ if its value at each diagonal point (m, m) lies in the order d subspace $\mathrm{Cliff}_d(T_m M) \subseteq \mathrm{Cliff}(T_m M)$.

Definition

Let $p \in \mathbb{Z}$. We shall say that a section σ of $S \boxtimes S^*$ over $M \times M$ has scaling order $\geq p$ (this is a type of vanishing order along the diagonal in $M \times M$) if

$$\mathrm{CliffordOrder}(D\sigma) \leq q - p$$

for every differential operator D of Getzler order $\leq q$.

Module of Regular Sections

Definition

Denote by $S(\mathbb{T}M)$ the space of all Laurent polynomials

$$\sigma = \sum_p \sigma_p t^{-p}$$

where σ_p is a smooth section of $S \boxtimes S^*$ over $M \times M$ with scaling order p or higher.

Lemma

$S(\mathbb{T}M)$ is a module over $A(\mathbb{T}M)$. □

Theorem

$S(\mathbb{T}M)$ generates a locally free sheaf over the sheaf of smooth functions $\mathbb{T}M$, and so determines a vector bundle $\boldsymbol{S}$ over $\mathbb{T}M$.

Fibers of the Rescaled Spinor Bundle

For $t \neq 0$ the fibers are

$$\varepsilon_{(m_1,m_2,t)} \colon \boldsymbol{S}_{(m_1,m_2,t)} \xrightarrow{\cong} S_{m_1} \otimes S^*_{m_2}$$

$$\varepsilon_{(m_1,m_2,t)} \colon \sum \sigma_p t^{-p} \longmapsto \sum \sigma_p(m_1, m_2) t^{-p}$$

(with apologies for the careless notation).

When $t = 0$ and $X_m = 0$ the fiber is

$$\varepsilon_{(0_m,t)} \colon \boldsymbol{S}_{(0_m,0)} \xrightarrow{\cong} \Lambda^* T_m M$$

$$\varepsilon_{(0_m,0)} \colon \sum \sigma_p t^{-p} \longmapsto \sum \mathrm{symbol}_p \, \sigma_p(m, m).$$

Note that the value $\sigma_p(m, m)$ lies in the order p part of $\mathrm{Cliff}(T_m M)$.

Fibers of the Rescaled Spinor Bundle, Continued

For general $X_m \in T_m M$ the formula

$$\boldsymbol{\nabla_X} \colon S(\mathbb{T}M) \longrightarrow S(\mathbb{T}M)$$
$$\boldsymbol{\nabla_X} \colon \sum \sigma_p t^{-p} \longmapsto \sum (\nabla_X \sigma_p) t^{-(p-1)}$$

induces an isomorphism

$$\exp(\boldsymbol{\nabla_X}) \colon \boldsymbol{S_{(X_m,0)}} \xrightarrow{\cong} \boldsymbol{S_{(0_m,0)}}.$$

So the restriction of $\boldsymbol{S}$ to $t{=}0$ is the pullback of $\Lambda^* TM$ to TM.

Simple, but:

$$\boxed{\exp(\boldsymbol{\nabla_X})\exp(\boldsymbol{\nabla_Y}) = \exp(\tfrac{1}{2}\boldsymbol{K(X,Y)})\exp(\boldsymbol{\nabla_{X+Y}})}$$

Model Operators and Getzler's Symbol

Theorem

If D is a linear partial differential operator on M, acting on the sections of S, and if D has Getzler-order no more than q, then the operators

$$D_{(m,\lambda)} = t^q D \qquad (t \neq 0)$$

extends to a smooth family of operators on the source-fibers of $\mathbb{T}M$, acting on the sections of the smooth vector bundle $\mathbb{S}$.

Theorem

When $D = \nabla_X$,

$$(D_{(m,0)} f)(Y_m) = (\partial_{X_m} f)(Y_m) + \tfrac{1}{2}\kappa(Y_m, X_m) \wedge f(Y_m).$$

Here $\kappa(Y_m, X_m)$ is the curvature of ∇, viewed in $\Lambda^2 T_m M$.

The Dirac Laplacian

When $\not{D}$ is the Dirac operator, which has Getzler order 2,

$$\not{D}_{(m,0)} = \text{de Rham differential on } T_m M.$$

The *square* of the Dirac operator *also* has Getzler order 2 and

$$\not{D}^2 = -\sum_a \nabla_{X_a} \nabla_{X_a} \text{ to leading Getzler order.}$$

The model operators for the square are therefore computable from the previous theorem:

$$(\not{D}^2)_{(m,0)} = -\sum_a \Big(\frac{\partial}{\partial x_a} + \tfrac{1}{2}\kappa(X_a, X_b) x_b \Big)^2$$

Rescaled Bundle—Five Minute University Version

A rescaled spinor bundle may be built over the tangent groupoid of a spin manifold M in much the same was as the tangent groupoid is itself built.

The construction uses Clifford algebra order as well as the Getzler order of differential operators on the spinor bundle of M.

The rescaled bundle leads to a new notion of model operator (or Getzler symbol). The Getzler model operators for the Dirac Laplacian encode the components Riemann curvature tensor.

Getzler's proof of the index theorem is encoded in the existence of a smooth, one-parameter family of supertraces on the convolution algebra of smooth sections.

Multiplicative Structure

There is a natural multiplication operation on the fibers of the rescaled spinor bundle $\boldsymbol{S}$ over $\mathbb{T}M$, at least away from $t = 0$:

$$\boldsymbol{S}_{(m_1, m_2, t)} \otimes \boldsymbol{S}_{(m_2, m_3, t)} \longrightarrow \boldsymbol{S}_{(m_1, m_2, t)}$$

since the above is nothing more than

$$S_{m_1} \otimes S^*_{m_2} \otimes S_{m_2} \otimes S^*_{m_3} \longrightarrow S_{m_1} \otimes S^*_{m_3}$$

Theorem

This extends smoothly to

$$\boldsymbol{S}_{(X_m, 0)} \otimes \boldsymbol{S}_{(Y_m, 0)} \longrightarrow \boldsymbol{S}_{(X_m + Y_m, 0)},$$

where the formula for the product is

$$\alpha \otimes \beta \longmapsto \exp(-\tfrac{1}{2}\kappa(X_m, Y_m)) \wedge \alpha \wedge \beta.$$

Convolution on the Tangent Groupoid

Let $\mathbb{G}$ be any Lie groupoid. Connes introduced and studied the following convolution product on $C_c^\infty(\mathbb{G})$:

$$f_1 \star f_2 \colon \gamma \longmapsto \int_{\gamma_1 \circ \gamma_2 = \gamma} f_1(\gamma_1) f_2(\gamma_2)$$

This extends immediately to $C_c^\infty(\mathbb{T}M, \boldsymbol{S})$. For $t \neq 0$ there is a restriction morphism

$$\varepsilon_t \colon C_c^\infty(\mathbb{T}M, \boldsymbol{S}) \longrightarrow \mathfrak{K}^\infty(L^2(M, S))$$

and for $t = 0$ there is a restriction morphism

$$\varepsilon_0 \colon C_c^\infty(\mathbb{T}M, \boldsymbol{S}) \longrightarrow C_c^\infty(TM, \Lambda^* TM),$$

where on the right the product is twisted convolution.

Traces on the Groupoid Algebra

The tangent groupoid algebra $C_c^\infty(\mathbb{T}M)$ carries a family of traces, parametrized by $t \neq 0$, obtained from the usual operator trace:

$$C_c^\infty(\mathbb{T}M) \xrightarrow{\varepsilon_t} \mathfrak{K}^\infty(L^2(M)) \xrightarrow{\mathrm{Tr}} \mathbb{C}.$$

Roughly speaking, local, or algebraic, index theory is the study of these traces as $t \to 0$.

The traces do not converge as $t \to 0$.

Instead more elaborate strategies must be developed, for instance replacing the traces with equivalent cyclic cocycles.

Supertraces on the groupoid algebra

Definition

Define

$$\int : C_c^\infty(TM, \Lambda^* TM)) \longrightarrow \mathbb{C}$$

by restriction to the zero section, followed by integration of the top-degree component over M.

Theorem (Index Theorem Without an Operator)

The supertraces

$$C_c^\infty(\mathbb{T}M, \mathbb{S}) \xrightarrow{\varepsilon_t} \mathfrak{K}^\infty(L^2(M, S)) \xrightarrow{\mathrm{Str}} \mathbb{C}$$

extend smoothly to $(2/i)^{\dim(M)/2}$ *times the supertrace*

$$C_c^\infty(\mathbb{T}M, \mathbb{S}) \xrightarrow{\varepsilon_0} C_c^\infty(TM, \wedge^* TM)) \xrightarrow{\int} \mathbb{C}$$

at $t = 0$.

Thank you!

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