Analytic Pontryagin Duality

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Objective

Let X be a smooth compact manifold. The Universal Coefficient Theorem for K-theory with coefficients in $\mathbb{R}/\mathbb{Z}$ asserts that there is an isomorphism

$$K^0(X, \mathbb{R}/\mathbb{Z}) \to \mathrm{Hom}(K_0(X), \mathbb{R}/\mathbb{Z}).$$

Using a geometric model of $K^0(X, \mathbb{R}/\mathbb{Z})$ and $K_0(X)$, we study an explicit ***analytic*** pairing implementing this map of the following form

$$\underbrace{\bar{\eta}(\partial\!\!\!/)}_{\text{analytic term}} \quad - \quad \underbrace{\int_M x \bmod \mathbb{Z}}_{\text{topological term}}.$$

The group $K^0(X, \mathbb{R}/\mathbb{Z})$

Definition ($\mathbb{R}/\mathbb{Z}$ K^0-cocycles)

Let X be a smooth compact manifold. Define a $\mathbb{R}/\mathbb{Z}$ K^0-cocycle over X as a triple

$$(g, (d, g^{-1}dg), \mu)$$

where

- $g : X \to U(N)$ is a smooth map, i.e. a K^1-representative of X ;
- $(d, g^{-1}dg)$ is a pair of flat connections on the trivial bundle τ ;
- $\mu \in \Omega^{\text{even}}(X)/\text{im}(d)$ satisfying

$$d\mu = ch(g, d) - \text{Tr}(g^{-1}dg)$$

Here, $ch(g, d)$ is the odd Chern character defined by

$$ch(g, d) = \sum_{n=0}^{\infty} \frac{n!}{(2n+1)!} \text{Tr}(g^{-1}dg)^{2n+1}.$$

Definition ($\mathbb{R}/\mathbb{Z}$ K^0-relation)

Let $\mathcal{E}_i = (g_i, (d, g_i^{-1}dg_i), \mu_i)$ where $g_i : X \to U(N_i)$, for $i = 1, 2, 3$. The $\mathbb{R}/\mathbb{Z}$ K^0-relation is given by $\mathcal{E}_2 = \mathcal{E}_1 + \mathcal{E}_3$ whenever we have $g_2 \simeq g_1 \oplus g_3$ (i.e. g_2 is homotopic to the unitary matrix $\mathrm{Diag}(g_1, g_3)$) and

$$\mu_2 = \mu_1 + \mu_3 - \mathrm{T}ch(g_1, g_2, g_3).$$

Here, $\mathrm{T}ch(g_1, g_2, g_3)$ denotes the transgression form of the odd Chern character which satisfies

$$d\mathrm{T}ch(g_1, g_2, g_3) = ch(g_1) - ch(g_2) + ch(g_3).$$

Definition ($\mathbb{R}/\mathbb{Z}$ K^0-group)

The group $K^0(X, \mathbb{R}/\mathbb{Z})$ consists of all $\mathbb{R}/\mathbb{Z}$ K^0-cocycles with zero virtual trace modulo the $\mathbb{R}/\mathbb{Z}$ K^0-relation.

Baum-Douglas Geometric K-homology

Let X be a smooth compact manifold. The (even) Baum-Douglas geometric K-homology $K_0(X)$ of X is a group generated by geometric K-cycles

$$(M, E, f)$$

where

- M is an even-dimensional smooth closed Spin^c-manifold, E is a complex vector bundle over M and $f : M \to X$ is a smooth map

modulo the following relation

- Direct sum-disjoint union :

$$(M, E_1 \oplus E_2, f) \sim (M, E_1, f) \sqcup (M, E_2, f)$$

- Bordism : $\exists (W, F, \varphi)$ s.t.

$$(\partial W, F_{|\partial W}, \varphi_{|\partial W}) \sim (M_1, E_1, f_1) \sqcup (-M_2, E_2, f_2)$$

- Vector bundle modification :

$$(M, E, f) \sim (\Sigma H, \beta_H \otimes \rho^* E, f \circ \rho)$$

Analytic term : the Dai-Zhang eta-invariant

Let (M, E, f) be an even K-cycle over X. Let $h = g \circ f : M \to U(N)$ be a K^1-representative of M.

- Consider the twisted Dirac operator $\partial\!\!\!/_{E\otimes\tau,M}$ acting on $L^2(S\otimes E\otimes\tau)$. Extend the bundle data trivially to the cylinder $M \times [0, 1]$.

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- Consider the twisted Dirac operator $\not{\partial}_{E\otimes\tau,M}$ acting on $L^2(S\otimes E\otimes\tau)$. Extend the bundle data trivially to the cylinder $M \times [0, 1]$.
- Over $M \times [0, 1]$ consider the Dirac-type operator

$$\not{\partial}^{\psi,h}_{E\otimes\tau,M\times[0,1]} = \not{\partial}_{E\otimes\tau} + (1-\psi)h^{-1}[\not{\partial}_{E\otimes\tau}, h].$$

Let $P^{\partial} = P_{>0,M} + P_{\mathcal{L}} : L^2(S \otimes E \otimes \tau|_M) \to L^2_{>0}(S \otimes E \otimes \tau|_M) \oplus \mathcal{L}$ be the modified APS projection, where $\mathcal{L} \in \mathrm{Lag}(\ker(\not{\partial}^{E\otimes\tau}_M))$.

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- Equip at both ends respectively with the boundary conditions

$$\begin{cases} P^{\partial} & \text{on } M \times \{0\} \\ \mathrm{Id} - h^{-1}P^{\partial}h & \text{on } M \times \{1\}. \end{cases}$$

  Then, $(\not{\partial}^{\psi,h}_{E\otimes\tau,M\times[0,1]}, P^{\partial}, \mathrm{Id} - h^{-1}P^{\partial}h)$ forms an elliptic self-adjoint boundary problem.

Its eta function is defined by

$$\eta(\not{\partial}^{\psi,h}_{E\otimes\tau,M\times[0,1]}, s) = \sum_{\lambda\neq 0,\lambda\in \text{spec}(\not{\partial})} \frac{\text{sgn}(\lambda)}{|\lambda|^s}$$

for $\text{Re}(s) >> 0$. Take $\eta(\not{\partial}^{\psi,h}_{E\otimes\tau,M\times[0,1]}) := \eta(\not{\partial}^{\psi,h}_{E\otimes\tau,M\times[0,1]}, 0)$.

Its eta function is defined by

$$\eta(\partial\!\!\!/^{\psi,h}_{E\otimes\tau,M\times[0,1]}, s) = \sum_{\lambda\neq 0, \lambda\in \mathrm{spec}(\partial\!\!\!/)} \frac{\mathrm{sgn}(\lambda)}{|\lambda|^s}$$

for $\mathrm{Re}(s) >> 0$. Take $\eta(\partial\!\!\!/^{\psi,h}_{E\otimes\tau,M\times[0,1]}) := \eta(\partial\!\!\!/^{\psi,h}_{E\otimes\tau,M\times[0,1]}, 0)$.

The Dai-Zhang eta-invariant

$$\hat{\eta}(M, E, h) = \tilde{\eta}(\partial\!\!\!/^{\psi,h}_{E\otimes\tau,M\times[0,1]}) - \mathrm{sf}\{\partial\!\!\!/^{\psi,h}_{E\otimes\tau,M\times[0,1]}(t)\}_{t\in[0,1]}$$

- $\tilde{\eta}(\partial\!\!\!/^{\psi,h}_{E\otimes\tau,M\times[0,1]}) = \frac{1}{2}(\dim\ker \partial\!\!\!/^{\psi,h}_{E\otimes\tau,M\times[0,1]} + \eta(\partial\!\!\!/^{\psi,h}_{E\otimes\tau,M\times[0,1]}))$.
- sf denotes the spectral flow of $\partial\!\!\!/^{\psi,h}_{E\otimes\tau,M\times[0,1]}(t)$ for $t \in [0, 1]$.

In our case, we take

$$\bar{\eta}_{DZ}(\partial\!\!\!/^{\psi,h}_{E\otimes\tau,M\times[0,1]}) \equiv \hat{\eta}(M, E, h) \bmod \mathbb{Z}.$$

Analytic pairing $K_0(X) \times K^0(X, \mathbb{R}/\mathbb{Z})$

Theorem

Let M be an even dimensional closed Spin^c manifold, $E \to M$ be a complex vector bundle. Let X be a smooth compact manifold, with $f : M \to X$ a smooth map. Let τ be the trivial bundle acted on by $h = g \circ f : M \to U(N)$. Let $\not{\partial}^{\psi,h}_{E\otimes\tau, M\times[0,1]}$ be the Dirac-type operator twisted by E and τ extended on the cylinder $M \times [0,1]$, defined by

$$\not{\partial}^{\psi,h}_{E\otimes\tau, M\times[0,1]} = \not{\partial}_{E\otimes\tau} + (1-\psi)h^{-1}[\not{\partial}_{E\otimes\tau}, h].$$

Let $\bar{\eta}_{DZ}(\not{\partial}^{\psi,h}_{E\otimes\tau, M\times[0,1]})$ be its reduced eta-invariant. Then, the analytic pairing $K_0(X) \times K^0(X, \mathbb{R}/\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ given by

$$\begin{aligned}\langle (M, E, f), (g, (d, g^{-1}dg), \mu)\rangle \\ = \bar{\eta}_{DZ}\left(\not{\partial}^{\psi,h}_{E\otimes\tau, M\times[0,1]}\right) - \int_M f^*\mu \wedge ch(E) \wedge Td(M) \bmod \mathbb{Z}\end{aligned}$$

is well-defined and non-degenerate.

Proof Sketch.

For the well-definedness, we verify that the analytic pairing formula

- is independent of the Riemannian metric of the manifold M and Hermitian metric and connection on E; (consider the cylinder connecting $M_i = (M, g_i)$ and (E_i, ∇^{E_i}), then take the difference of the two pairings and apply the Dai-Zhang Toeplitz index theorem on cylinder and Stokes theorem)
- respects the Baum-Douglas K-homology relations ; (For vector bundle modification, the non-trivial step is to show that the equality

$$\bar{\eta}_{DZ}(M, E, f) = \bar{\eta}_{DZ}(\Sigma H, \beta_{\Sigma H} \otimes \rho^* E, f \circ \rho)$$

  holds.)
- respects the $\mathbb{R}/\mathbb{Z}$ K^0-relation.

Continued.

Idea : apply the argument of the Mayer-Vietoris sequence in K-theory and the Five lemma for the non-degeneracy. Let $X = U \cup V$, consider

$$\begin{array}{ccccccc}
\longrightarrow K_c^0(U \cap V, \mathbb{R}/\mathbb{Z}) & \longrightarrow & K_c^0(U, \mathbb{R}/\mathbb{Z}) \oplus K_c^0(V, \mathbb{R}/\mathbb{Z}) & \longrightarrow & K^0(U \cup V, \mathbb{R}/\mathbb{Z}) \longrightarrow \\
\downarrow & & \downarrow & & \downarrow \\
\to \mathrm{Hom}(K_0(U \cap V), \mathbb{R}/\mathbb{Z}) & \to & \mathrm{Hom}(K_0(U), \mathbb{R}/\mathbb{Z}) \oplus \mathrm{Hom}(K_0(V), \mathbb{R}/\mathbb{Z}) & \to & \mathrm{Hom}(K_0(U \cup V), \mathbb{R}/\mathbb{Z}) \to
\end{array}$$

Continued.

Idea : apply the argument of the Mayer-Vietoris sequence in K-theory and the Five lemma for the non-degeneracy. Let $X = U \cup V$, consider

$$\begin{array}{ccccccc}
\longrightarrow K^0_c(U\cap V,\mathbb{R}/\mathbb{Z}) & \longrightarrow & K^0_c(U,\mathbb{R}/\mathbb{Z}) \oplus K^0_c(V,\mathbb{R}/\mathbb{Z}) & \longrightarrow & K^0(U\cup V,\mathbb{R}/\mathbb{Z}) \longrightarrow \\
\downarrow & & \downarrow & & \downarrow \\
\to \mathrm{Hom}(K_0(U\cap V),\mathbb{R}/\mathbb{Z}) & \to & \mathrm{Hom}(K_0(U),\mathbb{R}/\mathbb{Z}) \oplus \mathrm{Hom}(K_0(V),\mathbb{R}/\mathbb{Z}) & \to & \mathrm{Hom}(K_0(U\cup V),\mathbb{R}/\mathbb{Z}) \to
\end{array}$$

- It suffices to consider the case of $U \cong \mathbb{R}^n$ for even $n \in \mathbb{Z}_+$.
- The relevant K-homology is $K_0(\mathbb{R}^n) \cong \widetilde{K}_0(S^n)$, generated by $(S^n, \beta, \mathrm{Id})$, where β is the Bott bundle.
- We verify that the map $\widetilde{K}^0(S^n, \mathbb{R}/\mathbb{Z}) \to \mathrm{Hom}(\widetilde{K}_0(S^n), \mathbb{R}/\mathbb{Z})$ implemented by

$$\bar{\eta}(\partial\!\!\!/^{\beta}_{S^n\times[0,1]}) - \int_{S^n} (\mu - ch(\beta)) \wedge ch(\beta) \wedge \mathrm{Td}(S^n) \bmod \mathbb{Z}$$

  is an isomorphism.
- Apply induction on the size of open covers.

Analytic Pairing $H_2(X) \times H^2(X, \mathbb{R}/\mathbb{Z})$

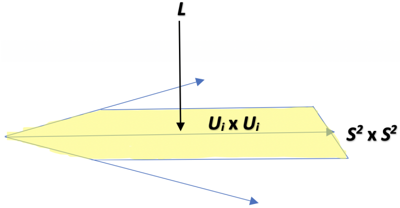
- Using the identification $H_2(X) \cong \Omega_2^{or}(X)$, we can consider a representative in $H_2(X)$ as $[S^2 \xrightarrow{f} X]$.
- The pairing reduces to $H_2(S^2) \times H^2(S^2, \mathbb{R}/\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$.
- Not clear how to twist a Dirac operator by a (flat) gerbe with connection.

Lemma

An alternative description of $H^2(S^2, \mathbb{R}/\mathbb{Z})$ is given by

$$H^2(S^2, \mathbb{R}/\mathbb{Z}) \cong H^2(S^2, \mathbb{R})/\widetilde{H}^2(S^2, \mathbb{Z})$$

whose representative is a pure Hermitan local line bundle L *ala* Melrose [10], with the curvature $B/2\pi$.



(a)

Together with a smooth composition isomorphism over $N_\epsilon(\mathrm{Diag}_3)$

$$H : \pi_3^* L \otimes \pi_1^* L \xrightarrow{\sim} \pi_2^* L.$$

$L \cong \mathcal{L}|_{U_i \times \{p_i\}} \otimes \mathcal{L}^{-1}|_{\{p_i\} \times U_i}$ over $U_i \times U_i$ using H.
mult. Herm. struc.

$$\langle u, v \rangle = \sum_i (\rho_i \times \rho_i) \langle u, v \rangle_i$$

- L is only defined locally on some neighbourhood of the diagonal of S^2. It has a multiplicative unitary connection ∇^L, compatible with the multiplication Hermitian structure.
- The appropriate Dirac operator is the projective Dirac operator $\eth^{L,\pm}_{S^2,\mathrm{proj}} \in \mathrm{Diff}^1(S^2; \mathcal{S}^\pm \otimes L, \mathcal{S}^\mp \otimes L)$, introduced by Mathai, Melrose and Singer [5,6]. It is an elliptic projective differential operator given by

$$\eth^{L}_{S^2,\mathrm{proj}} := cl \cdot \nabla^{\mathcal{S} \otimes L}_{\mathrm{left}}(\kappa_{\mathrm{Id}})$$

whose kernel is supported within the nbhd where L exists.

Subtleties

- $\partial\!\!\!/^{L}_{S^2,\text{proj}}$ does not have a spectrum –> $\bar{\eta}$?
- S^2 is even dimensional –> projective Dai-Zhang eta-invariant ?

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Definition

$$\eta_{DZ}\left(\partial\!\!\!/^L_{S^2,\mathrm{proj}}\right) := \eta_{APS}\left(\partial\!\!\!/^L_{S^2\times S^1,\mathrm{proj}}\right).$$

To calculate the RHS, consider the sharp product

$$\partial\!\!\!/^L_{S^2\times S^1,\mathrm{proj}} = \partial\!\!\!/^L_{S^2,\mathrm{proj}} \# \partial\!\!\!/_{S^1} = \begin{pmatrix} \partial\!\!\!/_{S^1}\otimes 1 & 1\otimes\partial\!\!\!/^{L,-}_{S^2,\mathrm{proj}} \\ 1\otimes\partial\!\!\!/^{L,+}_{S^2,\mathrm{proj}} & -\partial\!\!\!/_{S^1}\otimes 1 \end{pmatrix}.$$

The operator $\partial\!\!\!/^L_{S^2\times S^1,\mathrm{proj}}$ is elliptic and self-adjoint, it is still projective and does not have a spectrum. So, we define

Definition

$$\eta_{APS}\left(\partial\!\!\!/^L_{S^2\times S^1,\mathrm{proj}}\right) := \mathrm{Ind}\left(\partial\!\!\!/^{L,+}_{S^2,\mathrm{proj}}\right)\cdot\eta_{APS}\left(\partial\!\!\!/_{S^1}\right).$$

- $\mathrm{Ind}(\not{\partial}^{L,+}_{S^2,\mathrm{proj}}) = \mathrm{Tr}(\not{\partial}^{L,+}_{S^2,\mathrm{proj}} Q - 1) - \mathrm{Tr}(Q \not{\partial}^{L,+}_{S^2,\mathrm{proj}} - 1)$ is well-defined.
- $\eta_{APS}(\not{\partial}_{S^1})$ is the usual APS eta-invariant.

- $\mathrm{Ind}(\partial\!\!\!/_{S^2,\mathrm{proj}}^{L,+}) = \mathrm{Tr}(\partial\!\!\!/_{S^2,\mathrm{proj}}^{L,+} Q - 1) - \mathrm{Tr}(Q\partial\!\!\!/_{S^2,\mathrm{proj}}^{L,+} - 1)$ is well-defined.
- $\eta_{APS}(\partial\!\!\!/_{S^1})$ is the usual APS eta-invariant.

Since $\eta_{APS}(\partial\!\!\!/_{S^1}) = 0$, so is $\eta_{APS}\left(\partial\!\!\!/_{S^2\times S^1,\mathrm{proj}}^{L}\right)$. On the other hand, due to projectiveness, kernel of $\partial\!\!\!/_{S^2\times S^1,\mathrm{proj}}^{L}$ is not well-defined.

Assumption

$h(\partial\!\!\!/_{S^2\times S^1,\mathrm{proj}}^{L}) = \dim\ker(\partial\!\!\!/_{S^1}) = 1.$

Its reduced eta-invariant is

$$\bar{\eta}_{APS}(\partial\!\!\!/_{S^2\times S^1,\mathrm{proj}}^{L}) = \frac{\eta(\partial\!\!\!/_{S^2\times S^1,\mathrm{proj}}^{L}) + h(\partial\!\!\!/_{S^2\times S^1,\mathrm{proj}}^{L})}{2} \bmod \mathbb{Z} = \frac{1}{2} \bmod \mathbb{Z}.$$

- $\text{Ind}(\partial\!\!\!/^{L,+}_{S^2,\text{proj}}) = \text{Tr}(\partial\!\!\!/^{L,+}_{S^2,\text{proj}} Q - 1) - \text{Tr}(Q\partial\!\!\!/^{L,+}_{S^2,\text{proj}} - 1)$ is well-defined.
- $\eta_{APS}(\partial\!\!\!/_{S^1})$ is the usual APS eta-invariant.

Since $\eta_{APS}(\partial\!\!\!/_{S^1}) = 0$, so is $\eta_{APS}\big(\partial\!\!\!/^{L}_{S^2\times S^1,\text{proj}}\big)$. On the other hand, due to projectiveness, kernel of $\partial\!\!\!/^{L}_{S^2\times S^1,\text{proj}}$ is not well-defined.

Assumption

$h(\partial\!\!\!/^{L}_{S^2\times S^1,\text{proj}}) = \dim\ker(\partial\!\!\!/_{S^1}) = 1.$

Its reduced eta-invariant is

$$\bar{\eta}_{APS}(\partial\!\!\!/^{L}_{S^2\times S^1,\text{proj}}) = \frac{\eta(\partial\!\!\!/^{L}_{S^2\times S^1,\text{proj}}) + h(\partial\!\!\!/^{L}_{S^2\times S^1,\text{proj}})}{2} \mod \mathbb{Z} = \frac{1}{2} \mod \mathbb{Z}.$$

Then, the analytic PD pairing in H^2 is

$$\boxed{\bar{\eta}_{DZ}(\partial\!\!\!/^{L}_{S^2,\text{proj}}) - \int_{S^2} \frac{B}{2\pi} \mod \mathbb{Z}}.$$

where $B/2\pi$ is the first Chern class of L.

Quick Recap

- We propose a geometric model of the group $K^0(X, \mathbb{R}/\mathbb{Z})$ which can be incorporated into the construction of the Dai-Zhang eta-invariant.
- By pairing it with the usual even Baum-Douglas K-homology, we formulate an analytic pairing

$$\bar{\eta}_{DZ}\left(\eth^{\psi,h}_{E\otimes\tau_h, M\times[0,1]}\right) - \int_M f^*\mu \wedge ch(E) \wedge Td(M) \text{ mod } \mathbb{Z}$$

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  which is well-defined and non-degenerate.
- This is a direct (even) analog to Lott's K^1-pairing [6], thus closes the gap. As a result, it is reasonable to believe that this analytic K^0 pairing formula describes the Aharonov-Bohm effect of D-branes in Type-IIB String theory.
- We study one special case of the analytic Pontryagin duality pairing in $H^2(X, \mathbb{R}/\mathbb{Z})$ in which projective Dirac operators come into the picture.

Thank you for listening !

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Appendix

Analytic Pontryagin duality in H^1

Analytic Pairing $H_1(X) \times H^1(X, \mathbb{R}/\mathbb{Z})$

Classical pairing

$$H_1(X) \times H^1(X, \mathbb{R}/\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$$
$$([S^1 \xrightarrow{\gamma} X], A) \mapsto \int_{S^1} \gamma^* A \bmod \mathbb{Z}.$$

Analytic description

Let $\partial\!\!\!/_{S^1}$ be the usual Dirac operator on the circle S^1

$$\partial\!\!\!/_{S^1} = -i\frac{d}{d\theta}$$

whose kernel has dimension one. Let

$$L = \tilde{X} \times_\rho U(1)$$

be a flat line bundle over X, with connection ∇^L_A, where $\rho : \pi_1(X) \to U(1)$ is a unitary representation.

(Continued.)

Passing through $\gamma : S^1 \to X$ defines $(\tilde{L}, \nabla^{\tilde{L}})$ over S^1

$$\tilde{L} = \mathbb{R} \times_{\rho'} U(1), \quad \rho' = \gamma_* \circ \rho,$$

with a generating section $v_{\rho'}(\theta) = \exp(2\pi i\, a\, \theta)$, for $a \in (0, 1)$.

The twisted-by-$\tilde{L}$ Dirac operator $\partial\!\!\!/^{\tilde{L}}_{S^1}$ has eigenvalues $\lambda_n = n + a$. Its Atiyah-Patodi-Singer (APS) eta-invariant is $1 - 2a$. Then, the analytic pairing in H^1 is given by

$$\boxed{\bar{\eta}(\partial\!\!\!/^{\tilde{L}}_{S^1}) - \int_{S^1} \gamma^* A \bmod \mathbb{Z}.}$$

Remark

This is a special case of the analytic pairing in K^1 :

$$(S^1, \tau, \gamma) \in K_1(X), (L, \nabla^L, \omega) \in K^1(X, \mathbb{R}/\mathbb{Z})$$

with $d\omega = c_1(\nabla^L) = 0$.

The counterpart : Analytic K^1-pairing

Theorem

Let M be an odd dimensional closed Spin^c manifold, let E be a complex vector bundle over M. Let X be a smooth compact manifold with $f : M \to X$ a smooth map. Let $\partial\!\!\!/_M^{f^*V\otimes E}$ be the twisted Dirac operator, locally given by

$$\partial\!\!\!/_M^{f^*V\otimes E} = \sum_i c(e_i) \circ \nabla_{e_i}^{\mathcal{S}\otimes f^*V\otimes E}$$

for an o/n basis $\{e_i\}$. Then, the analytic pairing

$$K_1(X) \times K^1(X, \mathbb{R}/\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$$

given by (Lott [4])

$$\bar{\eta}_{APS}(\partial\!\!\!/_M^{f^*V\otimes E}) - \int_M f^*\omega \wedge ch(E) \wedge \mathrm{Td}(M) \text{ mod } \mathbb{Z}$$

is well-defined and non-degenerate.

Proof Sketch.

Idea : MV sequence in K^1 + the Five Lemma. Consider the part

$$\begin{array}{ccccccc} \longrightarrow K_c^1(U\cap V,\mathbb{R}/\mathbb{Z}) & \longrightarrow & K_c^1(U,\mathbb{R}/\mathbb{Z})\oplus K_c^1(V,\mathbb{R}/\mathbb{Z}) & \longrightarrow & K^1(U\cup V,\mathbb{R}/\mathbb{Z}) \longrightarrow \\ \downarrow & & \downarrow & & \downarrow \\ \to \mathrm{Hom}(K_1(U\cap V),\mathbb{R}/\mathbb{Z}) & \to & \mathrm{Hom}(K_1(U),\mathbb{R}/\mathbb{Z})\oplus \mathrm{Hom}(K_1(V),\mathbb{R}/\mathbb{Z}) & \to & \mathrm{Hom}(K_1(U\cup V),\mathbb{R}/\mathbb{Z}) \to \end{array}.$$

- Consider the case of $U \cong \mathbb{R}^n$ for odd $n \in \mathbb{Z}_+$. The relevant K-homology is $K_1(\mathbb{R}^n) \cong K_1(S^n)$, generated by (S^n, τ, Id).
- We verify that the map $K^1(S^n, \mathbb{R}/\mathbb{Z}) \to \mathrm{Hom}(K_1(S^n), \mathbb{R}/\mathbb{Z})$ implemented by

$$\bar{\eta}(\partial\!\!\!/_{S^n}) - \int_{S^n} \omega \wedge ch(\tau) \wedge \mathrm{Td}(S^n) \bmod \mathbb{Z}$$

  is an isomorphism.
- The topological term is dominated by $\mathrm{rk}(\tau) \int_{S^n} \omega \bmod \mathbb{Z}$.
- The reduced APS eta-invariant is given in the table below.

n	$\eta(\partial\!\!\!/_{S^n})$	$\dim\ker(\partial\!\!\!/_{S^n})$	$\bar{\eta}(\partial\!\!\!/_{S^n})$
1	0	1	$\frac{1}{2}$
≥ 3	0	0	0