The Dirac operator under collapse to a smooth manifold

Saskia Roos

University of Potsdam

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Gromov-Hausdorff convergence

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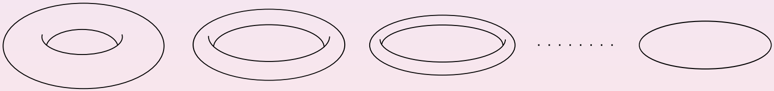
Any sequence $(M_i, g_i)_{i\in\mathbb{N}}$ contains a subsequence that converges with respect to the Gromov-Hausdorff distance to a compact metric space B.



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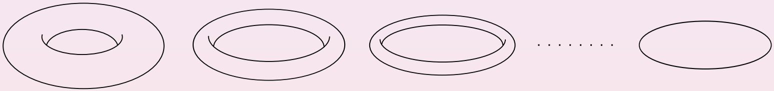

Assumption

The limit space B is a smooth manifold.

Structure of Collapse

Theorem (Cheeger-Fukaya-Gromov '92)

Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a convergent sequence in $\mathcal{M}(n, d)$ with a smooth limit space (B, h). Then for all i sufficiently large there are metrics $\tilde{g}_i$ on M_i and $\tilde{h}_i$ on B such that

$$\lim_{i\to\infty} \|\tilde{g}_i - g_i\|_{C^1} = 0, \qquad \lim_{i\to\infty} \|\tilde{h}_i - h\|_{C^1} = 0,$$

and $f_i : (M_i, \tilde{g}_i) \to (B, \tilde{h}_i)$ is a *Riemannian affine fiber bundle*,

The Dirac operator

Let (M, g) be a spin manifold and ΣM the spinor bundle.
The Dirac operator acts on spinors $\varphi \in \Gamma(\Sigma M)$ as

$$D\varphi = \sum_{i=1}^{n} \gamma(e_i)\nabla_{e_i}\varphi,$$

for a local orthonormal frame $(e_1, \ldots, e_n)$.

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Proposition (Lott '02)

Let (M, g) be a closed spin manifold and $(g_i)_{i\in\mathbb{N}}$ be a sequence of Riemannian metrics such that $\lim_{i\to\infty} \|g_i - g\|_{C^1} = 0$.
Then $\lim_{i\to\infty} \sigma(D_i) = \sigma(D)$.

Spin structures

Let $f : (M, g) \to (B, h)$ be a Riemannian affine fiber bundle and assume that (M, g) has a fixed spin structure.

There is an induced spin structure on each fiber $f^{-1}(p)$, $p \in B$. But there is no induced spin structure on B in general.

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Example

- $f : S^5 \to \mathbb{CP}^2$. Here $f^* w_2(\mathbb{CP}^2) = 0$, but $w_2(\mathbb{CP}^2) \neq 0$.
- $S^1 \times S^2 / \mathbb{Z}_2 \to \mathbb{RP}^2$, where $(x, y) \backsim (x', y')$ iff $x = \overline{x'}$, $y = -y'$.

Affine parallel spinors

The induced metrics on the fibers of a Riemannian affine fiber bundle $f : (M, g) \to (B, h)$ are parallel with respect to ∇^{aff}.
$\Longrightarrow$ There exists an induced connection ∇^{aff} on ΣM.

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Proposition (R. '18)

There exists a Clifford bundle $\mathcal{P}$ over B and an isometry

$$Q : S^{\mathrm{aff}} \to L^2(\mathcal{P}).$$

Previous results I

Theorem (Ammann '98)

Let $\left(f_i : (M_i, g_i) \to (B, h_i)\right)_{i\in\mathbb{N}}$ be a collapsing sequence of Riemannian S^1-bundles. Assume that (M_i, g_i) is a closed $(n+1)$-dimensional spin manifold.

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- $\|l_i F_i\|_\infty \xrightarrow{i\to\infty} 0$,
- $\limsup_{i\to\infty} \|\operatorname{grad} l_i\|_\infty < \frac{1}{2}$,

then $\lim_{i\to\infty} \sigma(D^{M_i}_{|(S_i^{\mathrm{aff}})^\perp}) = \{\pm\infty\}$,

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and $\sigma(D^{M_i}_{|S_i^{\text{aff}}})$ converges to $\sigma(D^B)$ if n is even and to $\sigma(D^B \oplus -D^B)$ if n is odd.

Previous results II

Theorem (Lott '02)

Let $\left(f_i : (M_i, g_i) \to (B, h_i)\right)_{i\in\mathbb{N}}$ be a collapsing sequence of Riemannian affine bundles. Assume that (M_i, g_i) is a spin manifold in $\mathcal{M}(n, d)$.

Then there exists a subsequence such that

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Above results also apply to the Hodge Dirac operator $\not{D} = \mathrm{d} + \mathrm{d}^*$ on differential forms.

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There exist also analogous results for singular limit spaces.

Example: Hodge Dirac vs. spin Dirac

Consider the collapsing sequence $(T^2, \mathrm{d}s^2 + \frac{1}{i^2} e^{2\cos(s)} \mathrm{d}t)_{i \in \mathbb{N}}$ with limit space $(S^1, \mathrm{d}s^2)$.

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Hodge Dirac:
Affine parallel forms: $S^{\mathrm{aff}} = \{f(s)\} \cup \{\alpha(s)\mathrm{d}(s)\}$.

$$\not{D}^i_{S^{\mathrm{aff}}}(f + \alpha \mathrm{d}s) = \frac{\partial f}{\partial s}\mathrm{d}s + \frac{\partial}{\partial s}\left(e^{\cos(s)}\alpha\right).$$

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Spin Dirac:
By Ammann's result: $\lim_{i\to\infty} \sigma(D^i_{S_i^{\mathrm{aff}}}) = \sigma(D^{S^1} \oplus -D^{S^1})$.

Main result

Theorem (R. '18)

Let $(f_i : (M_i, g_i) \to (B, h_i))_{i \in \mathbb{N}}$ be a collapsing sequence of Riemannian affine fiber bundles sucht that (M_i, g_i) is a spin manifold in $\mathcal{M}(n, d)$. Then there exists a Clifford bundle $\mathcal{P}$ over B and a first order elliptic self-adjoint differential operator

$$D_R^B = D^{T^\infty} + \gamma(Z_\infty) + \gamma(A_\infty)$$

with coefficients in $C^{0,\alpha}$ on $\mathcal{P}$ such that

$$\sigma(D^{M_i}_{|S_i^{\mathrm{aff}}}) \xrightarrow{i \to \infty} \sigma(D_R^B)$$

Special case

Corollary

If in addition,

- the holonomy of the vertical bundles is trivial,
- the instrinsic curvature of the fibers is flat,
- the horizontal distribution is integrable,

in the limit, then there is an induced spin structure on B and the spectrum of $\sigma(D^{M_i}_{|S_i^{\mathrm{aff}}})$ converges to the spectrum of

$$\begin{cases} D^B \oplus D^B, & \text{if } n \text{ is even and } \dim(B) \text{ is odd,} \\ D^B, & \text{else.} \end{cases}$$

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Main problem: The operators D^{M_i} are not defined on the same space.
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$\mathrm{Spin}(n) \curvearrowright L^2(P_i, g_i^P) \otimes \Sigma_n$ by isometries.

There is an isometry $\left(L^2(P_i, g_i^P) \otimes \Sigma_n\right)^{\mathrm{Spin}(n)} \to L^2(\Sigma M_i)$.

Sketch of the proof II

By an equivariant version of Gromov's compactness result, there is a convergent subsequence

$$(P_i, g_i^P) \xrightarrow{i\to\infty} (B^P, h^P)$$

such that (B^P, h^P) is a Riemannian manifold and $\mathrm{Spin}(n) \curvearrowright (B^P, h^P)$ by isometries.

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such that (B^P, h^P) is a Riemannian manifold and
$\mathrm{Spin}(n) \curvearrowright (B^P, h^P)$ by isometries.
By an equivariant version of Cheeger-Gromov-Fukaya's results, there are metrics h_i^P on B^P such that

$$\begin{array}{ccc} (P_i, g_i^P) & \xrightarrow{\tilde{f}_i} & (B^P, h_i^P) \\ \downarrow & & \downarrow \\ (M_i, g_i) & \xrightarrow{f_i} & (B, h_i), \end{array}$$

$\lim_{i \to \infty} \|h_i^P - h^P\|_{C^1} = 0$ and $\tilde{f}_i : P_i^P \to B^P$ is a Riemannian affine fiber bundle.

Sketch of the proof III

There is an isometry

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In total we obtain

$$\begin{array}{ccc}
\left(L^2(P_i, g_i^P)^{\mathrm{aff}} \otimes \Sigma_n\right)^{\mathrm{Spin}(n)} & \longrightarrow & S_i^{\mathrm{aff}} \\
\downarrow & & \downarrow Q_i \\
\left(L^2(B^P, h_i^P) \otimes \Sigma_n\right)^{\mathrm{Spin}(n)} & \longrightarrow & \left(L^2(B^P, h^P) \otimes \Sigma_n\right)^{\mathrm{Spin}(n)} \\
 & & \| \\
 & & L^2(\mathcal{P})
\end{array}$$

Sketch of the proof IV

Now $\left(Q_i \circ D^{M_i}_{S_i^{\text{aff}}} \circ Q_i^{-1}\right)_{i\in\mathbb{N}}$ is a sequence of operators on $\mathcal{P}$ and

$$Q_i \circ D^{M_i}_{S_i^{\text{aff}}} \circ Q_i^{-1} = D^{T_i} + \gamma(Z_i) + \gamma(A_i).$$

Sketch of the proof IV

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$$Q_i \circ D^{M_i}_{S_i^{\mathrm{aff}}} \circ Q_i^{-1} = D^{T_i} + \gamma(Z_i) + \gamma(A_i).$$

Proposition

The coefficients of D^{T_i}, Z_i and A_i are uniformly bounded in $C^1(B)$.

$\Rightarrow$ There is a subsequence such that the spectrum converges.

□

Example: Convergence to Dirac eigenvalues

Let (G, g) be a closed simply-connected n-dimensional Lie group with a biinvariant metric.
Fix a maximal torus T^k and consider $B := {}_{T^k}\backslash^{G}$ with the induced quotient metric h.

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Collapse along the T^k fibers

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Check:

- vertical bundle has trivial holonomy,
- fibers are embedded flat tori,
- horizontal distribution is integrable in the limits.

Thus, the $\sigma(D^i_{|S_i^{\mathrm{aff}}})$ converges to the spectrum of D^B, resp. $D^B \oplus -D^B$.

The end

Thank you!