Bardeen-Cooper-Schrieffer Theory of Superconductors on Hyperbolic Surfaces

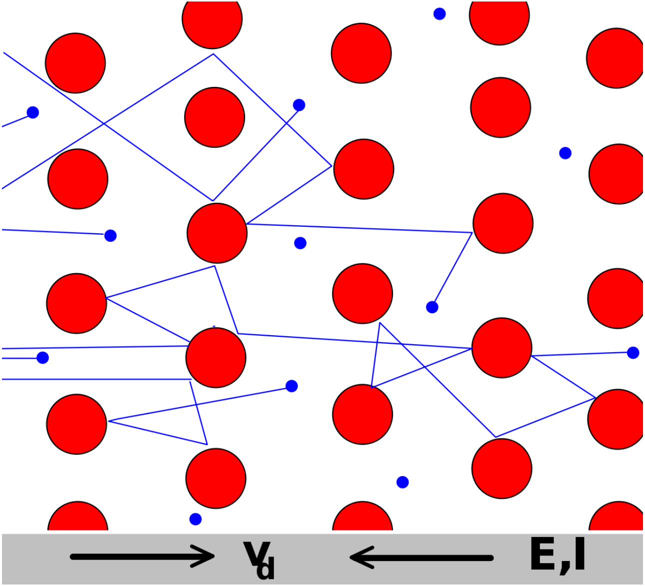
Ahnaf Tajwar Tahabub

Supervisors: Elder Professor and Australian Laureate Fellow Mathai Varghese
DECRA Fellow David Baraglia

University Adelaide, Australia

October 1, 2019

Intuition Behind BCS Theory



Modelling Physical Systems

Most natural (due to Jonathan J Gleason) definition of a physical system is

Definition

A physical system is a collection of observables $O(\mathcal{A})$ (self-adjoint elements) in a separable unital complex F^*-algebra $\mathcal{A}$.

A state is a positive normalised restricted linear functional $\omega : \mathcal{A} \to \mathbb{C}$ which assigns expectation values to observables

- Physical systems are thus treated using probabilities/statistics
- Values of observables are measured by machines which have restrictions $\rightsquigarrow$ reduces to C^* algebra
- A quantum physical system $\rightsquigarrow$ non-commutative $\mathcal{A}$
- A classical Physical system $\rightsquigarrow$ commutative $\mathcal{A}$

Hilbert-Space Representation

- For a complex Hilbert Space $L(H)$ denotes the set of closed densely defined linear operators.

Theorem

(Gelfand-Naimak for F^*-algebras) For every separable unital F^* algebra $\mathcal{A}$ there is a separable Hilbert Space H and isomorphism π from $\mathcal{A}$ to a separable unital F^*-subalgebra of $L(H)$.

For each ω state in $\mathcal{A}$ there is a unique positive unit trace operator $\rho : H \to H$ with $\omega(a) = Tr(\rho\pi(a)) =: \langle\pi(a)\rangle_\rho$

- Choosing a Hilbert Space is like choosing a frame to describe the system with observables inside $L(H)$ and states as unit trace positive operators ρ.

Pure and Mixed States

- A state $\rho : H \to H$ is like a probability measure
- Can compute Von Neumann Entropy (uncertainty prior to measurement) of state

$$S(\rho) = \mathrm{Tr}(\rho \ln \rho)$$

- Fact: $S(\rho) = 0 \iff \rho^2 = \rho \iff \rho = |\phi\rangle\langle\phi|$ for a normalised ϕ $\rightsquigarrow$ called pure states (usual QM)
- By spectral theorem, we can write $\rho = \sum_{n\in\mathbb{N}} c_n |\phi_n\rangle\langle\phi_n|$ where ϕ_n is an orthonormal basis and c_n are convex coefficients.
- So high entropy when states are more mixed i.e. c_n are uniformly distributed

Manifolds and Coordinates

- We each view the universe using a collection varying 3 real numbers for space and one for time, $(x_1, x_2, x_3, t) \rightsquigarrow$ coordinates
- Only things we agree (smoothly) on are considered to be real ...
- Smooth manifolds are precisely spaces which admit smoothly transitioning local coordinates and forms the global object that describes the universe as a collective of all perspectives.
- Assuming non-relativistic conditions we separate space and time to get space-time manifold $M \times \mathbb{R}$

Classification of Spaceforms

- Suppose the space manifold M is connected and carries a complete Riemannian metric with constant curvature $\rightsquigarrow$ spaceform

Theorem

(Killing-Hopf) Each spaceform M is isometric to a quotient of R^n (flat), a sphere (positive curvature) or hyperbolic space (negative curvature) by a discrete subgroup of isometries acting freely.

- We will focus on $n = 2$ and the negative curvature case i.e. $M = H^2/G$ where $G \leq Iso(H^2)$ discrete and acting freely $\rightsquigarrow$ Hyperbolic surface

Hyperbolic Space and Surfaces

- Hyperbolic space can be represented as upper half plane $HP = \{x + iy \in \mathbb{C} : y > 0\}$ with metric $g = (dx^2 + dy^2)/y^2$

  $\rightsquigarrow$ Geodesics are semi-circles perpendicular to $y = 0$ and vertical lines

  $\rightsquigarrow$ $\mathrm{Iso}^+(HP) \cong PSL(2, \mathbb{R})$, acting by Mobius transforms with $z \mapsto -\bar{z}$ (refection) to get orientation reversing ones

  $\rightsquigarrow$ $PSL(2, \mathbb{R})$ generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (rotation), $\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ (dilation) and $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ (translation)
- Discrete free subgroups consisting of dilations and translations give hyperbolic surfaces

Isometries of Hyperbolic Space

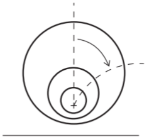


FIGURE 3. An elliptic transformation rotates hyperbolic circles around a fixed center.

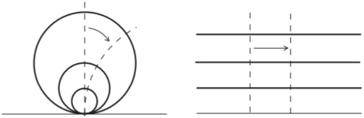


FIGURE 4. A parabolic transformation fixes a point on $\partial\mathbb{H}$.

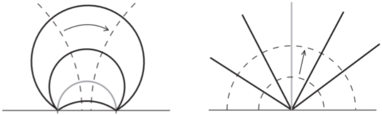


FIGURE 5. A hyperbolic transformation translates between two fixed points on $\partial\mathbb{H}$. The axis (unique fixed geodesic) is shown in gray.

Types of Particles: Electrons

- Elementary free particles are precisely the irreducible unitary representations of the Poincare Group on a Hilbert Space $\rightsquigarrow$ they are classified by their mass $m \geq 0$ and spin (or helicity) $s = 0, 1/2, 1, 3/2, \ldots$
- Integer spin $\rightsquigarrow$ Bosons   Half-Integer spin $\rightsquigarrow$ Fermions
- Experimentally all spin $1/2$ particles are massive and an example is the electron
- Electron couples with electromagnetic field (also weak and gravity) so they have electrical charge q.
- Technically 'electrons' don't exist on hyperbolic surfaces ...

First Quantisation of Electron

- Fix a complex hermitian line bundle $L \to M$ to incorporate coupling to $U(1)$ for electromagnetic field.

Definition

The Hilbert space $\mathcal{H}$ for a single electron's state on M are the square-integrable sections $L^2(M, L) \otimes \mathbb{C}^2$ where $\mathbb{C}^2$ is for spin $1/2$.

- If the line bundle is trivial, no vorticies and global coupling to external field then we get $\mathcal{H} \cong L^2(M) \otimes \mathbb{C}^2$
- Since $M = HP/G$, elements of $\mathcal{H}$ are precisely $\phi \in L^2(HP, L) \otimes \mathbb{C}^2$ with $\psi(gx) = \psi(x)$ for all $g \in G$ (invariant under G).

Second Quantisation Multi-Particle Systems

All particles are indistinguishable in quantum theory and so quantum states must retain probability upon exchange

Theorem

(Spin-Statistics) Fermions are anti-symmetric under exchange of particle exchange while Bosons are symmetric.

Definition

If the Hilbert space $\mathcal{H}$ represents a single particle then

1. n Fermion system has $\mathcal{H}_f^n = \wedge^n \mathcal{H}$ as Hilbert Space
2. n Boson system has $\mathcal{H}_b^n = S^n \mathcal{H}$ as Hilbert Space

Fock Space

If we allow particle numbers to change then we use the Full Fock Space

Definition

For an indefinite number of fermions we have Fermioninc Fock space $\mathcal{F}_{\mathcal{H}} = \bigoplus_{n=0}^{\infty} \mathcal{H}_f^n$ while Bosonic Fock Space is $\mathcal{B}_{\mathcal{H}} = \bigoplus_{n=0}^{\infty} \mathcal{H}_b^n$ where $\mathcal{H}_f^0 = \mathcal{H}_b^0 = \mathbb{C}\,\Omega$ where $\Omega \in \mathcal{H}$ is a normalised vacuum state (could be different for each).

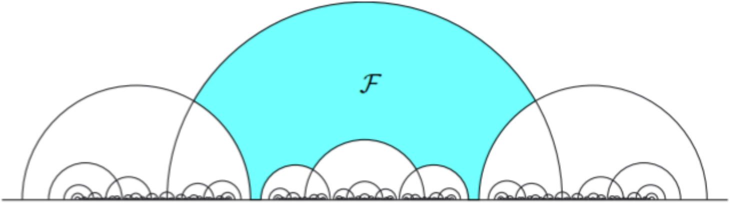
Definition

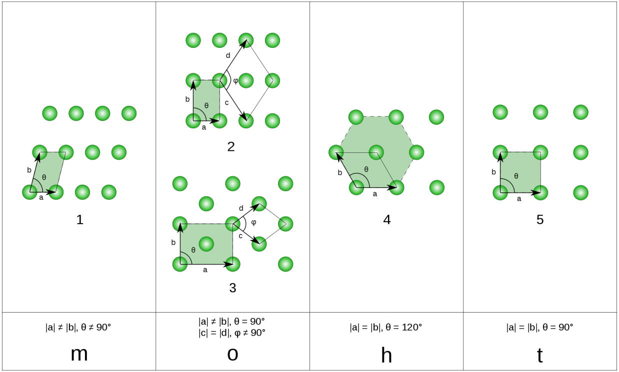
For $\phi \in \mathcal{H}$, we can define creation operator $a(\phi)^\dagger : \mathcal{F}_{\mathcal{H}} \to \mathcal{F}_{\mathcal{H}}$ which works by wedging and annihilation operator $a(\phi) : \mathcal{F}_{\mathcal{H}} \to \mathcal{F}_{\mathcal{H}}$ which works by contraction (these are adjoints)

Potential Describing Lattice V

- Treat the cation lattice as a potential (First quantisation)
- Lattice determined by Fuchsian group G since $M = HP/G$
- Suppose $M = HP/G$ is geometrically finite ($\iff$ finitely generated) so that it admits a finite side geodesic polygonal fundamental domain.
- So define $V \in L^1(M^2)$ invariant under geodesic flow between points.
- Understanding electron in fundamental domain, we can extend to whole system by group G
- Same as considering system on torus (periodic boundary conditions on square lattice).

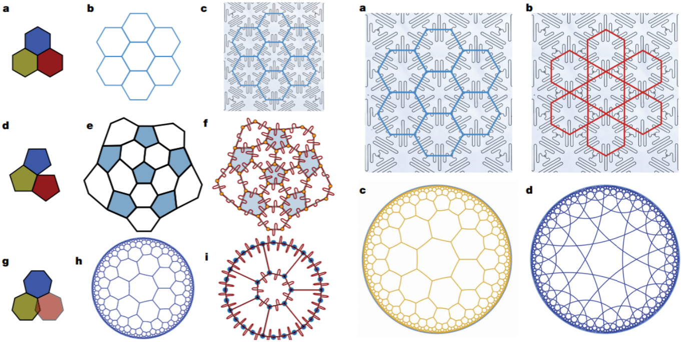
Finite Lattice





Experimental Hyperbolic Lattice

- The paper 'Hyperbolic lattices in circuit quantum electrodynamics' by Alicia J. Kollar, Mattias Fitzpatrick & Andrew A. Houck shows that we have constructed Hyperbolic Lattices in the real world.
- Using the disc model (isometric to half plane model) we have



Kinetic Energy Operator

- Suppose there is an external global magnetic field potential real one-form A and electric potential $W : M \to \mathbb{R}$
- This defines a connection $\nabla^A = i\ d + A$ on trivial line bundle with Laplacian Δ_A

Definition

A single particle kinetic energy operator is $T = \Delta_A + W$ where W acts by multiplication.

Hamiltonian and Number Operator

- Let ϕ_i be an orthonormal basis for $\mathcal{H}$ and consider $a_i^\dagger = a(\phi_i)^\dagger$ and $a_i = a(\phi_i)$.

Definition

The Hamiltonian on the Fermionic Fock Space is

$$\mathbb{H} = \sum_{i,j} T_{ij} a_i^\dagger a_j + \frac{1}{2} \sum_{ijkl} V_{ijkl} a_i^\dagger a_j^\dagger a_l a_k$$

where $T_{ij} = \langle \phi_i, T\phi_j \rangle$ and $V_{ijkl} = \langle \phi_i \otimes \phi_j, V(\phi_k \otimes \phi_l) \rangle$ (V acts by multiplication).

Definition

The number operator is given by

$$N = \sum_i a_i^\dagger a_i$$

Landau Grand Potential Energy

Definition

For a state ρ of the Fermionic Fock Space $\mathcal{F}_{\mathcal{H}}$ we can define the Grand Potential Energy $\mathcal{F}$ at a temperature T and chemical potential μ to be

$$\mathcal{F}(\rho) = \langle \mathbb{H} \rangle_\rho - TS(\rho) - \mu \langle N \rangle_\rho = \mathrm{Tr}((\mathbb{H} - \mu N)\rho) - TS(\rho)$$

- The grand potential depends on volume V (from inner product and metric), temperature T and chemical potential μ as natural variables.

Theorem

Under constant V, T and μ, the equilibrium state ρ is the one that minimises $\mathcal{F}$

It is the Gibbs state which does this minimisation.

Quasi-Free or BCS States

The idea of BCS is that we want states which come in pairs of fermions only. These lead directly to quasi-free or BCS states

Definition

A quasi-free state ρ is one which has non-zero n point functions only for n even and all n-point functions are determined by 2-point functions.

$$\langle a_1^{\#} a_2^{\#} \ldots a_{2n}^{\#} \rangle = \sum_{\sigma \in S'_{2n}} (-1)^{\sigma} \langle a_{\sigma(1)}^{\#} a_{\sigma(2)}^{\#} \rangle_{\rho} \ldots \langle a_{\sigma(2n-1)}^{\#} a_{\sigma(2n)}^{\#} \rangle_{\rho}$$

$$\langle a_1^{\#} \ldots a_{2n+1}^{\#} \rangle_{\rho} = 0$$

Here $\#$ is nothing or and $a_j^{\#} = a(f_j)^{\#}$ for $f_j \in \mathcal{H}$ and S'_{2n} is the subset of S_{2n} with $\sigma(1) < \sigma(3) \ldots < \sigma(2n-1)$ and $\sigma(2j-1) < \sigma(2j)$ for $1 \leq j \leq n$

Generalised One-Particle Density Operator

Definition

For a quasi-free state ρ there is a generalised one-particle density operator $\Gamma : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ which completely determines ρ given by

$$\langle (\phi_1, \phi_2), \Gamma(\psi_1, \psi_2) \rangle = \langle [a^\dagger(\psi_1) + a(\overline{\psi}_1][a(\phi_1) + a^\dagger(\overline{\phi_2})] \rangle_\rho$$

- There are operators $\gamma, \alpha : \mathcal{H} \to \mathcal{H}$ so that we have 2×2 matrix

$$\Gamma = \begin{pmatrix} \gamma & \alpha \\ \alpha^\dagger & 1 - \overline{\gamma} \end{pmatrix}$$

- γ is the one-particle density, α is the pairing excitation
- Solving for ρ is the same as solving for (γ, α).

Assuming $SU(2)$ invariance for Spin

- We want to get rid of spin dependence on the functional so we suppose that $S^\dagger \gamma S = \gamma$ and $S^\dagger \alpha \overline{S} = \alpha$ for all $S \in SU(2)$.
- This means precisely that $\gamma = \tilde{\gamma} \otimes I$ and $\alpha = \tilde{\alpha} \otimes \sigma_2$
- $\tilde{\gamma}$ and $\tilde{\alpha}$ are now operators on $L^2(M)$ without tensoring with $\mathbb{C}^2$
- The same spin-independence is true for kernels.

BCS Functional

Using quasi-free states which are spin independent

Definition

In terms of (γ, α) with $\gamma(x, y)$ and $\alpha(x, y)$ the spin independent integral kernals of operators, we have

$$
\begin{aligned}
\mathcal{F}_{BCS}(\gamma, \alpha) = {} & 2\mathrm{Tr}_{\mathcal{H}}(\Delta_A + \mu + W)\gamma - 2TS(\Gamma) \\
& + \int_{M \times M} |\alpha(x, y)|^2 V(x, y) dg(x) dg(y) \\
& - \int_{M \times M} |\gamma(x, y)|^2 V(x, y) dg(x) dg(y) \\
& + 2 \int_{M \times M} \gamma(x, x)\gamma(y, y) V(x, y) dg(x) dg(y)
\end{aligned}
$$

Discussion of BCS Functional

- α is a measure of the pairing while γ looks at individual particles
- It can be shown that the *BCS* functional always attains a minimum under mild assumptions.
- When $\alpha \neq 0$ we have a superconducting state while if $\alpha = 0$ then we have a normal state
- The temperature at which a the minimiser changes from normal to superconducting is called the critical temperature T_c.
- An expression for this can be establised spectrum of a certain Hessian matrix.

Ginzburg-Landau Functional

- Phenomenological theory of superconductivity near critical temperature using a first quantised section ϕ of $L^2(M, L)$ or in our simpler case, $L^2(M)$.
- Here ϕ represents the wave function for the Bosonic Cooper pairs with the modulus $|\phi(x)|^2$ the density of pairs
- Expanding potential upto second order in density (fourth order in $|\phi|$), in presence of external electric and magnetic potentials W and A give

Definition

The Ginzurg-Landau Functional for $\phi \in L^2(M)$ is

$$\mathcal{F}_{GL}(\phi) = \int_M |\nabla^A \phi|^2 + \lambda_1 W|\phi|^2 - \lambda_2 D|\phi|^2 + \lambda_3|\phi|^4 dg$$

where λ_i depend on microscopic parameters while D depends on the temperature i.e. $T = T_c(1 - Dh^2)$ with h small.

Reduction of BCS into Ginzburg-Landau

- After assuming a certain kind of translation invariance we have the main theorem

Theorem

Let M be a compact hyperbolic surface and $\mathcal{H} = L^2(M) \otimes \mathbb{C}^2$ and fix external fields A and W. For $T = T_c(1 - Dh^2)$ there is a λ_0 and $\lambda_1, \lambda_2, \lambda_3$ (in GL functional) giving

$$\inf_{\Gamma} \mathcal{F}_{BCS}(\Gamma) = \mathcal{F}(\Gamma_0) + \lambda_0 h \inf_{\phi} \mathcal{F}_{GL}(\phi) + o(h)$$

where Γ_0 is a normal state with $\alpha = 0$.

- One can think of GL as the first derivative (linear approximation) of *BCS* at the critical temperature T_c

Non-conventional and High Temperature Superconductors

- Superconductors with T_c higher than $30K$ cannot be explained using BCS theory
- Need a microscopic theory for these even in flat space
- Can be used for better Maglev trains and efficient nuclear fusion!
- Can use methods of string theory i.e. AdS/CFT to do this ...