Edge-following topological states

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Analysis on Manifolds

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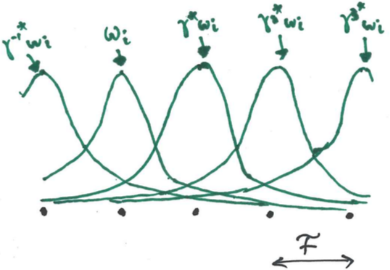
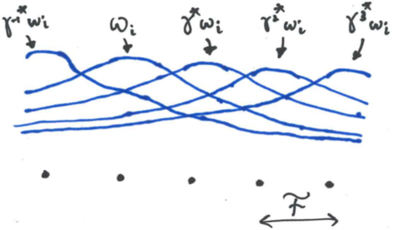
[1] based on arXiv:1908.09559, and 19xx.xxxxx (with M. Ludewig)

How do you see a topological phase of matter??

Lattice Hilbert space $\leftrightarrow$ some copies of $\ell^2_{\mathrm{reg}}(\mathbb{Z}^d)$ inside $L^2(\mathbb{R}^d)$.





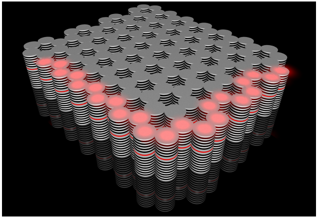
Topological quantum chemistry: a topological insulator has spectral subspaces which are "bad" copies of $\ell^2_{\mathrm{reg}}(\mathbb{Z}^d)$.
Maths: non-free Hilbert $C^*_r(\mathbb{Z}^d)$-modules [Ludewig+T, 1904.13051].

These abstract characteristics are mostly *invisible*! So what exactly do physicists see?

"Topological physics" on the edge

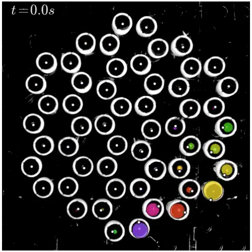
In the last five years, physicists have successfully realised Chern topological insulators in photonics, acoustics, cold atoms, metamaterials, Floquet systems, exiton-polaritons...

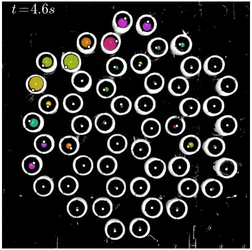
A Chern insulator is a 2D material, described in the idealised boundaryless-limit by a $\mathbb{Z}^2$-invariant Hamiltonian operator $H = H^*$ on $\ell^2_{\mathrm{reg}}(\mathbb{Z}^2) \otimes \mathbb{C}^2$ having a remarkable kind of spectral gap.

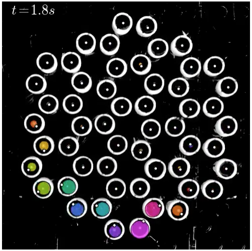


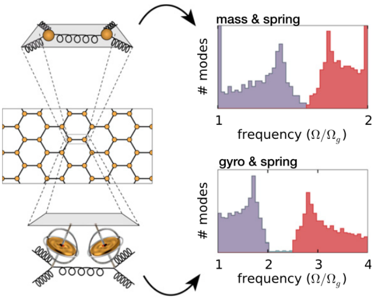
When the (material) boundary is introduced, the spectral gap of H is completely filled up with edge-following topological states!

Experiments[2]: edge-following states



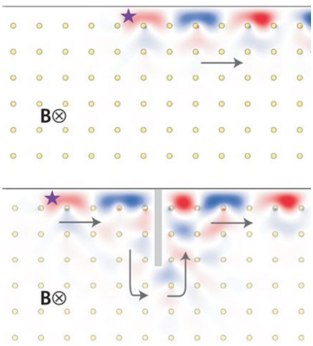


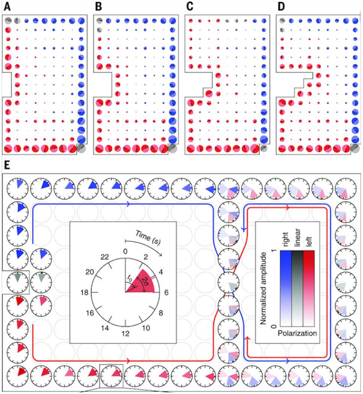


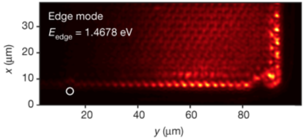


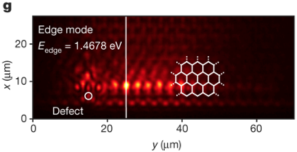
[2] Nash et al, PNAS (2015)

Experiments[3]: edge-following states









[3] Lu et at, Nature Photonics (2014); Süsstrunk, Huber, Science (2015); Klembt et at, Nature (2018)

Some history

Quantum Hall effect (1980) $\overset{\text{convoluted}}{\rightsquigarrow}$ Chern insulator Hamiltonians on lattice Hilbert space $\ell^2_{\text{reg}}(\mathbb{Z}^2) \otimes \mathbb{C}^2$:

$$H_{\text{Chern example}} = \begin{pmatrix} m + U_x + U_y & -iU_x - U_y \\ -iU_x + U_y & -m - U_x - U_y^* \end{pmatrix} + \text{adjoint}.$$

Here U_x, U_y are unit translations in x and y directions. For $0 < m < 2$, this has a spectral gap and realises a "Chern insulator".

I will prove directly that *any* Chern insulator *must* acquire crazy edge-following states which fill up spectral gap.

Hope: motivate mathematical investigation[4] into general bulk-edge correspondences, especially coarse index perspective.

[4] Prior work is geometrically limited to very special straight edges.

Abstract Chern insulator

Regular representation: $\mathbb{Z}^2 \ni \gamma \mapsto U_\gamma \in \mathcal{B}(\ell^2_{\rm reg}(\mathbb{Z}^2))$.
These operators generate the reduced group C^*-algebra $C^*_r(\mathbb{Z}^2)$.

Generic translation invariant Hamiltonian:

$$H = H^* = \sum_{\gamma \in \mathbb{Z}^2} U_\gamma \otimes W_\gamma \in \mathcal{B}(\overbrace{\ell^2_{\rm reg}(\mathbb{Z}^2) \otimes \mathbb{C}^2}^{2 \text{ d.o.f. / site}}),$$

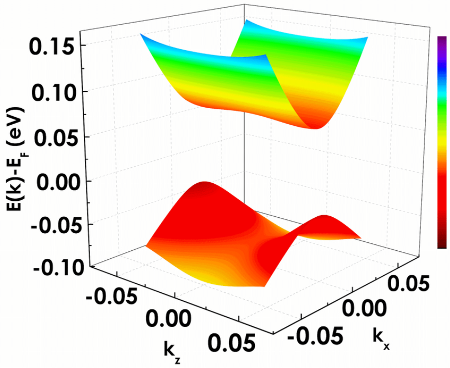
with each $W_\gamma = W^*_{-\gamma}$ a 2×2 *hopping matrix*.

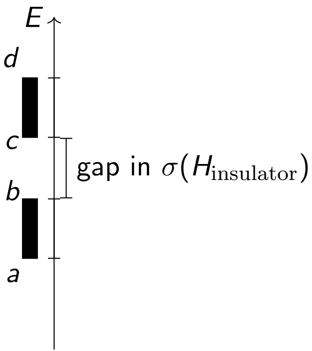
Locality: Sufficiently fast decay of $\gamma \mapsto W_\gamma \Rightarrow$

$$H \in M_2(C^*_r(\mathbb{Z}^2)) \overset{\text{Fourier}}{\cong} C(\mathbb{T}^2; M_2(\mathbb{C})).$$

After Fourier transform, H becomes a continuous family $\{H_k\}_{k \in \mathbb{T}^2}$ of 2×2 Hermitian matrices, acting on two copies of $L^2(\mathbb{T}^2)$.

Abstract Chern insulator

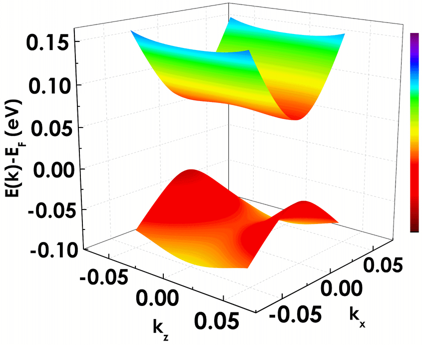


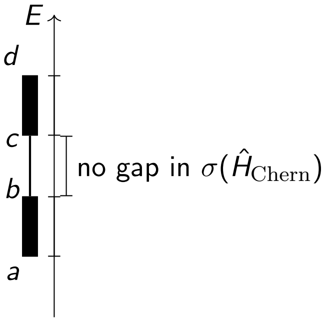


Each $H_k, k \in \mathbb{T}^2$, has two eigenvalues, and $\sigma(H) = \cup_{k\in\mathbb{T}^2}\, \sigma(H_k)$.
Defn: $H = H_{\text{insulator}}$ spectrum comprises two separated bands.

Eigenspaces for lower energy band form a line bundle $\mathcal{L}_{\text{low}} \to \mathbb{T}^2$, classified by first Chern class in $H^2(\mathbb{T}^2, \mathbb{Z}) \cong \mathbb{Z}$. For $H_{\text{Chern example}}$, get the generating class!

Abstract Chern insulator





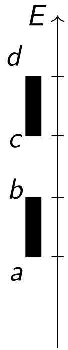
Definition: A *Chern insulator* H_{Chern} is a $H_{\mathrm{insulator}}$ such that $c_1(\mathcal{L}_{\mathrm{low}})$ is the generator $\underbrace{[\mathfrak{b}]}_{\mathrm{Bott}}$ of $H^2(\mathbb{T}^2) \cong \mathbb{Z} \cong \widetilde{K}^0(\mathbb{T}^2)$.

Physics observation: Let S be lattice points lying on one side of a partition (i.e. in the material sample). Truncated $\hat{H}_{\mathrm{Chern}}$ acting on $\ell^2(S) \otimes \mathbb{C}^2$ acquires spectra filling up the gap of H_{Chern}!

Abstract Chern insulator

Unlike *idealised* H, the *true* truncated Hamiltonians $\hat{H}$ do not enjoy $\mathbb{Z}^2$ symmetry, and Fourier transform fails.

Nevertheless, with C^*-algebras, can relate the spectra of H and $\hat{H}$!



Recall that $H = H^* \in M_2(C_r^*(\mathbb{Z}^2))$. For $H_{\text{insulator}}$, spectral gap gives room for the lower band spectral projection to be given by continuous functional calculus:

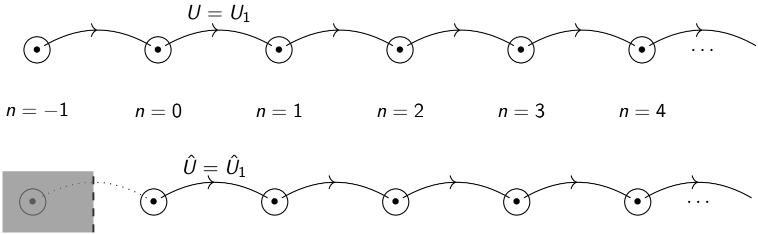
$$\varphi(H_{\text{insulator}}) \in M_2(C_r^*(\mathbb{Z}^2)), \qquad \varphi(\lambda) = \begin{cases} 1, & \lambda \in [a, b] \\ 0, & \lambda \geq c. \end{cases}$$

In K-theory: $[\varphi(H_{\text{Chern}})] = [\mathfrak{b}] \in K_0(C_r^*(\mathbb{Z}^2)) = K^0(\mathbb{T}^2)$.

Preliminaries: Toeplitz algebra

Instead of U_γ, truncated Hamiltonians $\hat{H}$ live in "Toeplitzified" version of $C^*_r(\mathbb{Z}^2)$ generated from truncated translations $\hat{U}_\gamma$.

1D Example: If $\mathbb{Z}$ is visualised on a line, what happens to U_γ upon truncation to the right half-line: $\ell^2(\mathbb{Z}) \to \ell^2(\mathbb{N})$?



Generating translation $U = U_1$ becomes the unilateral shift $\hat{U}$, which is a non-unitary isometry with index -1.

Preliminaries: Toeplitz algebra

Define $C_r^*(\mathbb{N})$ to be the C^*-subalgebra of $\mathcal{B}(\ell^2(\mathbb{N}))$ generated by $\hat{U}$.

Think of $C_r^*(\mathbb{N})$ as a "quantisation" of $C_r^*(\mathbb{Z})$ taking U to $\hat{U}$.
Symbol homomorphism $\pi : C_r^*(\mathbb{Z}) \to C_r^*(\mathbb{N})$ takes $\hat{U}$ back to U.

Observation: The boundary projection $p_{n=0} = 1 - \hat{U}\hat{U}^*$ is killed by π, and generates the compact operator ideal $\mathcal{K}(\ell^2(\mathbb{N}))$.

$$\text{Short exact sequence} \quad \boxed{0 \to \mathcal{K} \to C_r^*(\mathbb{N}) \xrightarrow{\pi} C_r^*(\mathbb{Z}) \to 0}$$

The invertible operator $U \in C_r^*(\mathbb{Z})$ lifts *incurably* to a non-invertible $\hat{U}$ in $C_r^*(\mathbb{N})$: there is a *topological* obstruction!

Preliminaries: Toeplitz algebra

More precisely, any invertible function $f \in C(\mathbb{T}) \cong C_r^*(\mathbb{Z})$ lifts to a Toeplitz operator $T_f \in C_r^*(\mathbb{N})$ which is Fredholm.
Any other lift T_f+compact has the same Fredholm index.

Toeplitz index theorem [F. Noether '21]

Non-invertibility of T_f, as measured by analytic Fredholm index, actually equals the topological winding number index of f.

Homological algebra: lifting obstructions in SES can be detected by connecting maps, but these are hard to compute.

K-theory is powered by Bott periodicity: LES $\rightsquigarrow$ cyclic 6-term exact sequences $\Rightarrow$ much better chance of being computable!

K-theory for operator algebras

For $\mathcal{A}$ a unital C^*-algebra, $K_0(\mathcal{A})$ is Grothendieck group of isomorphism classes of projections in $M_\infty(\mathcal{A}) = \lim_{N\to\infty} M_N(\mathcal{A})$.

$K_1(\mathcal{A})$ is homotopy classes of unitaries in $\mathcal{U}_\infty(\mathcal{A})^+$.

Example: $K_0(\mathcal{K}) \overset{\text{Morita}}{\cong} K_0(\mathbb{C}) \cong K^0(\mathrm{pt}) \cong \mathbb{Z}$, and $K_1(\mathcal{K}) = 0$.

Example: $K_0(C_r^*(\mathbb{Z})) = K^0_{\mathrm{top}}(\mathbb{T}) \cong \mathbb{Z}$ generated by identity projection/trivial line bundle.

Example: $K_1(C_r^*(\mathbb{Z})) = K_1(C(\mathbb{T})) \cong \mathbb{Z}$ generated by U, or the basic Laurent polynomial $z \mapsto z$ with winding number 1.

Example: $K_0(C_r^*(\mathbb{Z}^2)) = K^0_{\mathrm{top}}(\mathbb{T}^2) \cong \mathbb{Z} \oplus \mathbb{Z}$ generated by trivial and Bott line bundles.

K-theoretic Toeplitz index

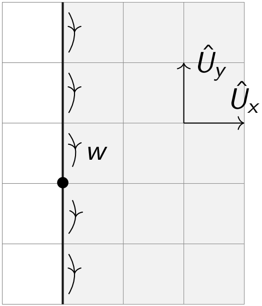
K-theory turns $\boxed{0 \to \mathcal{K} \to C_r^*(\mathbb{N}) \xrightarrow{\pi} C_r^*(\mathbb{Z}) \cong C(\mathbb{T}) \to 0}$ into

$$\begin{array}{ccccc}
\overbrace{K_0(\mathcal{K})}^{\mathbb{Z}} & \longrightarrow & \overbrace{K_0(C_r^*(\mathbb{N}))}^{\mathbb{Z}} & \longrightarrow & \overbrace{K_0(C(\mathbb{T}))}^{\mathbb{Z}} \\
\mathrm{Ind}\Big\uparrow & & & & \Big\downarrow\mathrm{Exp} \\
\underbrace{K_1(C(\mathbb{T}))}_{\mathbb{Z}} & \longleftarrow & \underbrace{K_1(C_r^*(\mathbb{N}))}_{0} & \xleftarrow{0} & \underbrace{K_1(\mathcal{K})}_{0}
\end{array}$$

$\hat{U} \in C_r^*(\mathbb{N})$ has index -1. So Ind is an isomorphism, and the middle two groups are solved.

Exp is a suspended/ "higher" index composed with Bott isomorphism $K_2 \cong K_0$; measures obstruction to lifting *projections*.

2D version: Half-plane Toeplitz algebra



Original $C^*_r(\mathbb{Z}^2)$ was generated by commuting unitaries U_x and U_y.

Toeplitzification means truncating $\ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{N} \times \mathbb{Z})$;
Get *isometries* $\hat{U}_x$, $\hat{U}_y$ generating the **semigroup algebra** $C^*_r(\mathbb{N} \times \mathbb{Z})$.

$$\text{SES} \qquad \boxed{0 \to \mathcal{I} \to C^*_r(\mathbb{N} \times \mathbb{Z}) \xrightarrow{\pi} C^*_r(\mathbb{Z}^2) \to 0}.$$

Kernel $\mathcal{I}$ is generated by edge-projection $P_{x=0} = 1 - \hat{U}_x \hat{U}_x^*$.

Observation: the edge-travelling operator $w = \hat{U}_y^* P_{x=0} \in \mathcal{I}$.
We will see that w is the smoking gun of edge-following states!

LES for half-plane algebra (Künneth)

$$
\begin{array}{ccccc}
\overbrace{K_0(\mathcal{I})}^{\mathbb{Z}} & \xrightarrow{0} & \overbrace{K_0(C^*_r(\mathbb{N}\times\mathbb{Z}))}^{\mathbb{Z}[1]} & \xrightarrow{\pi_*} & \overbrace{K_0(C^*_r(\mathbb{Z}^2))}^{\mathbb{Z}[1]\oplus\mathbb{Z}[\mathfrak{b}]} \\
\uparrow \text{Ind} & & & & \downarrow \text{Exp} \\
\underbrace{K_1(C^*_r(\mathbb{Z}^2))}_{\mathbb{Z}^2} & \xleftarrow{\pi_*} & \underbrace{K_1(C^*_r(\mathbb{N}\times\mathbb{Z}))}_{\mathbb{Z}} & \xleftarrow{0} & \underbrace{K_1(\mathcal{I})}_{\mathbb{Z}[w]}
\end{array}
$$

The LES yields $\boxed{[w] = \mathrm{Exp}[\mathfrak{b}] \equiv \mathrm{Exp}[\varphi(H_{\mathrm{Chern}})]}$.

Slogan: *When performing half-space truncation, obstruction to maintaining spectral gap of a Chern insulator is the edge-travelling operator!*

Gap-filling phenomenon [T, 1908.05995]

Just as $\varphi(H_{\mathrm{Chern}}) \in M_2(C^*_r(\mathbb{Z}^2))$, also $\varphi(\hat{H}_{\mathrm{Chern}}) \in M_2(C^*_r(\mathbb{N} \times \mathbb{Z}))$.
While the former is a projection,

Theorem

$\varphi(\hat{H}_{\mathrm{Chern}}) \in M_2(C^*_r(\mathbb{N} \times \mathbb{Z}))$ *is no longer a projection.*

Proof.

Otherwise, $\varphi(\hat{H}_{\mathrm{Chern}})$ gives a class in $K_0(C^*_r(\mathbb{N} \times \mathbb{Z}))$, and

$$\begin{aligned} 0 \overset{\mathrm{exact}}{=} \mathrm{Exp}(\pi_*[\varphi(\hat{H}_{\mathrm{Chern}})]) &= \mathrm{Exp}[\varphi(\pi(\hat{H}_{\mathrm{Chern}}))] \\ &= \mathrm{Exp}[\varphi(H_{\mathrm{Chern}})] \\ &= \mathrm{Exp}[\mathfrak{b}] = [w] \neq 0 \in K_1(\mathcal{I}). \end{aligned}$$

□

Gap-filling phenomenon [T, 1908.05995]

Corollary

$\hat{H}_{\mathrm{Chern}}$ has spectrum filling the entire gap (b,c) in $\sigma(H_{\mathrm{Chern}})$.

Proof.

Choose $\mathrm{supp}(\varphi') = [b', c']$ with $b \le b' < c' \le c$ arbitrarily.
Since $\varphi(\hat{H}_{\mathrm{Chern}})$ is not a projection,

$$\emptyset \neq \{\lambda \in \sigma(\hat{H}_{\mathrm{Chern}}) \,:\, \varphi(\lambda) \neq 0, 1\} \subset [b', c'].$$

□

Remark: Same argument for $\hat{H}_{\mathrm{Chern}} + (\text{pert. in } \mathcal{I})$.
Remark: Exp was first exploited by Kellendonk–Richter–Schulz-Baldes to understand quantised edge conductance in QHE.

Gap-filling states ⇒ quantised boundary currents

Connes' cyclic cohomology gives a NC version of de Rham currents.

Example: $U \in C^*_r(\mathbb{Z})$ Fourier transforms to Laurent $z : e^{i\theta} \mapsto e^{i\theta}$.

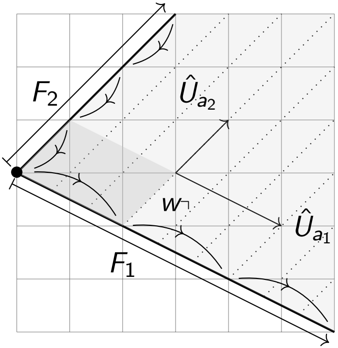
$$\langle \mathrm{Wind}, [z] \rangle = \frac{1}{2\pi i} \int_{\mathbb{T}} z^{-1} dz \quad \in \mathbb{Z} \subset \mathbb{C}$$

pairs a cyclic 1-cocycle integrally with $[z] \in K_1(C^\infty(\mathbb{T}))$.

Sketch for $[w] = \mathrm{Exp}[\varphi(H_{\mathrm{Chern}})]$:
Let X be position-along-boundary-operator:

$$1 = \underbrace{\tau}_{\frac{\text{trace}}{\text{length}}}(w^{-1}[X, w]) = \tau(\underbrace{\varphi'(H_{\mathrm{Chern}})}_{\substack{\text{gap's density} \\ \text{matrix}}} \underbrace{\dot{X}}_{\text{velocity}}) = \text{edge current}$$

Propagating around corners



Corner of a material $\approx$ convex cone $\Rightarrow$ subsemigroup $S \subset \mathbb{Z}^2$ preserves truncation.

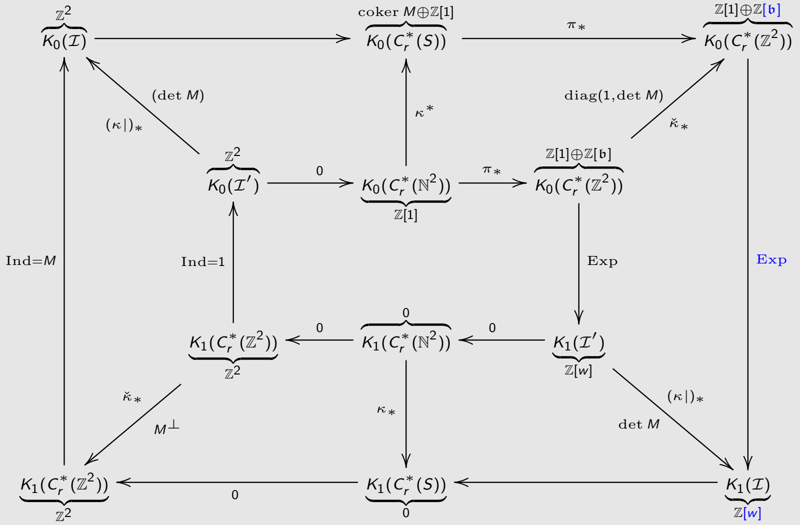
$$0 \to \mathcal{I} \to C_r^*(S) \xrightarrow{\pi} C_r^*(\mathbb{Z}^2) \to 0.$$

Compute whether $\mathrm{Exp}[\mathfrak{b}] \neq 0 \in K_1(\mathcal{I})$. If so, conclude that $\hat{H}_{\mathrm{Chern}}$ acquires gap-filling spectra.

Observation: $\ker(\pi) = \mathcal{I}$ is generated by face projections P_{F_1}, P_{F_2}.
Observation: Edge-travelling operator $w_\neg = \hat{U}_{a_2}^* P_{F_2} + \hat{U}_{a_1} P_{F_1} \in \mathcal{I}$.
Observation: There exists an index 1 Fredholm operator in $C_r^*(S)$.

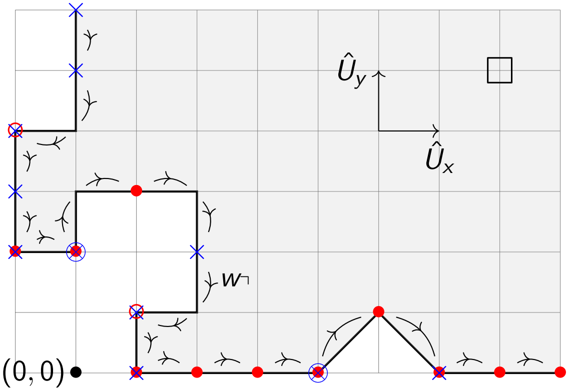
Propagating around corners

Theorem (T, 1908.05995; cf. Cuntz '17)



Bumpy corners

Harder analysis, but slogan still true: $[\varphi_{H_{\mathrm{Chern}}}] \equiv [\mathfrak{b}] \overset{\mathrm{Exp}}{\mapsto} [w_\neg]$.



Conclusion: Any physical realisation of abstract Chern insulator will have gap-filling and edge-following states that produce quantised boundary currents.

Partitioned manifold "coarse" index

This K-theory machine works much more generally: continuum version is in progress, with M. Ludewig. Lattice computation is "embedded", as in quantum chemistry.

Remark: In '88, J. Roe discovered a Partitioned manifold index theorem: Dirac operator on noncompact manifold X has a "coarse index" in $K_1(\underbrace{C^*(X)}_{\text{Roe algebra}})$, defined via an Exp map.

Compact partitioning hypersurface Y defines a cyclic 1-cocycle which eats this K_1-index to give a number equal to the index of associated Dirac on Y. "*Bulk index localises to boundary*"!

Example: $K_1(C^*(\text{line})) \cong \mathbb{Z}$, generated by the coarse index of $\mathrm{i}\frac{d}{dx}$. Alternatively, the edge-travelling operator is a generator!

Edge-following topological states: general phenomenon

Γ-invariant (magnetic) Hamiltonians H on $L^2(X)$ give spectral projections $\varphi(H)$ defining $K_0(C_r^*(\Gamma))$ classes (see [T+L 1904.13051])

One expects $[\varphi(H_{\rm top})]$ to be detected by truncating to $L^2(U \subset X)$ and looking for gap-filling states appearing at ∂U.

In analogous K-theory machine, $C^*(\partial U) \sim \mathcal{I}$, and indeed $\mathrm{Exp} : K_0(C_r^*(\Gamma)) \xrightarrow{\neq 0} K_1(C^*(\partial U))$ in examples.

In $\dim(X) = 2$, typically $\partial U \sim_{\rm coarse}$ line, then
Then $K_1(C^*(\partial U)) \cong \mathbb{Z}$, with generator an "edge-travelling operator" w hopping along a discretisation of ∂U, contributing one unit of "edge current". Thus $\mathrm{Exp}[\varphi(H_{\rm topological})]$ counts how many units of w the edge states of $\hat{H}_{\rm topological}$ are equivalent to.