An equivariant Atiyah-Patodi-Singer index theorem

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(Higher) APS index of D_h on manifold with boundary is a bridge between primary and secondary invariants.

Motivation and Questions

- X: complete Riemannian manifold;
- D: Dirac type operator $D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}$, $(D^+)^* = D^-$;
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- (Gromov-Lawson) X spin having uniform positive scalar curvature outside a compact set M;
- (Atiyah-Patodi-Singer index) M is a compact manifold with boundary N where M has product metric near N and N has a psc metric. $X \backslash M = N \times [0, \infty)$.

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Near N, $D_M = \sigma(D_N - \frac{\partial}{\partial u})$ where

$$D_N^* = D_N : L^2(N, E^+) \to L^2(N, E^+)$$

and

$$\sigma : E^+|_N \to E^-|_N$$

is a bundle isomorphism.

Recall: Atiyah-Patodi-Singer index theorem

Denote by

$$P_{\geq 0} = \chi_{[0,\infty)}(D_N).$$

APS boundary condition: .

$$H^1(M, E^+, P) := \{\psi \in H^1(M, E^+) : P_{\geq 0}(\psi|_N) = 0\}.$$

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Theorem (Atiyah-Patodi-Singer)

$D_M^+ : H^1(M, E^+, P) \to L^2(M, E^-)$ *is a Fredholm operator with index*

$$\mathrm{ind}_{APS}\, D_M = \int_M \hat{A}(M)\mathrm{ch}(E/S) - \frac{\eta(D_N) + \dim\ker D_N}{2}$$

where

$$\eta(D_N) = \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr}(D_N e^{-t^2 D_N^2}) dt$$

is the eta invariant measuring the spectral asymmetry of D_N.

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Reference:

- P. Hochs, B-L Wang, **Wang**: arXiv 2019.

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Notation:

- D_M, D_C: restriction to M, $C := X \backslash M$ respectively.

Fredholm index as K-theoretic boundary map

Fact: If R is a parametrix for D^+, i.e.,
$1 - RD^+ = S_0, 1 - D^+R = S_1$ are compact, then

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In fact, the invertible element $\begin{bmatrix} 0 & R \\ D^+ & 0 \end{bmatrix}$ in $\mathcal{B}/\mathcal{K}$ can be lifted to an invertible element $L = \begin{bmatrix} S_0 & -(S_0+1)R \\ D^+ & S_1 \end{bmatrix} \in \mathcal{B}$.

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Fredholm index is given by the boundary map

$$\begin{aligned} K_1(\mathcal{B}/\mathcal{K}) &\to K_0(\mathcal{K}) \cong \mathbb{Z} \\ \begin{bmatrix} 0 & R \\ D^+ & 0 \end{bmatrix} &\mapsto L^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} L - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} S_0^2 & S_0(S_0+1)R \\ S_1 D^+ & 1 - S_1^2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &\mapsto \operatorname{Tr}(S_0^2) - \operatorname{Tr}(S_1^2) = \operatorname{Tr}(S_0) - \operatorname{Tr}(S_1). \end{aligned}$$

A parametrix

Choose $Q = \frac{1-e^{tD^-D^+}}{D^-D^+}D^+$ such that

$$\tilde{S}_0 := 1 - QD^+ = e^{-tD^-D^+} \qquad \tilde{S}_1 := 1 - D^+Q = e^{-tD^+D^-}.$$

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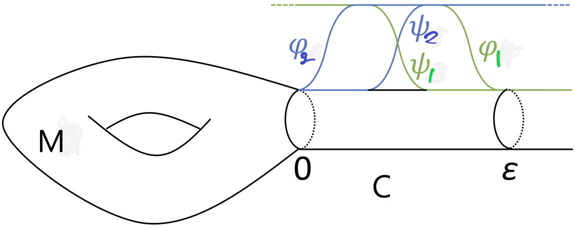
$$R = \phi_1 Q \psi_1 + \phi_2 Q_C \psi_2.$$



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S_0, S_1 are trace class operators with smooth kernels. R is a parametrix for D^+.

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Therefore,

$$\begin{aligned} \mathrm{ind} D =& \dim \ker D^+ - \dim \ker D^- \\ =& \mathrm{Tr}(S_0) - \mathrm{Tr}(S_1) \\ =& [\mathrm{Tr}(S_0') - \mathrm{Tr}(S_1)] + [\mathrm{Tr}(S_0) - \mathrm{Tr}(S_0')] \end{aligned}$$

where $S_0' := \psi_1 \tilde{S}_0 \phi_1 + \psi_1 Q \phi_1' + \psi_2 Q_C \phi_2'$.

Evaluation of Fredholm index

Proposition (Hochs-Wang-W)

As $t \to 0^+$,

$$\mathrm{Tr}(S_0') - \mathrm{Tr}(S_1) \to \int_M \hat{A}(X) \wedge \mathrm{ch}(E/S)$$

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Corollary (Hochs-Wang-W)

If M *is a compact manifold with boundary* N *and* $C = N \times [0,\infty)$ *is the cylindrical end, then*

$$\mathrm{ind}\, D = \int_M \hat{A}(M) \wedge \mathrm{ch}(E/S) - \frac{1}{2}\eta(D_N).$$

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- This strategy can be lifted to construct higher APS index and evaluation the equivariant APS-index.
- This method can be related to Melrose's b-calculus approach to APS index, which is lifted to define a geometric representative in the higher APS index for Galois covering (Leichtnam-Piazza).

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- Assume the boundary operator D_N to have isolated spectrum at 0.

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Remark

The APS boundary condition is replaced by the notion of spectral sections for the case of family (Melrose-Piazza) and Galois covers (Leichtnam-Piazza). For X, it is equivalent to a perturbation of D_N so it is invertible.

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Idea:

From Fredholm operator to general Fredholm operator:

- For compact Z, in $\mathcal{L}(L^2(Z))$, the ideal $\mathcal{K}(L^2(Z))$ is small;
- For noncompact X, "in $D^*(X)$, the ideal $C^*(X)$ is small".

Recall: Roe algebra

Let X be a manifold and H a Hilbert space with nondegenerate representation of $C_0(X)$.

- $T \in \mathcal{B}(H)$ is locally compact if $T\chi_K, \chi_K T \in \mathcal{K}(H)$ for any compact $K \subset X$;
- T has finite propagation if $\exists r > 0$ such that for $Y, Z \subset X$ we have $\chi_Y T \chi_Z = 0$ whenever $d(Y, Z) > r$.

Definition

- Roe algebra $C^*(X)$ is the norm closure of locally compact operators with finite propagation;

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Definition

- Roe algebra $C^*(X)$ is the norm closure of locally compact operators with finite propagation;
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- For closed $Y \subset X$, the relative Roe algebra $C^*(X, Y) \subset C^*(X)$ is the ideal generated by $C^*(Y)$.

Recall: Roe's localised coarse index

Let $Z \subset X$ be a closed subset and D a Dirac type operator on $E \to X$ ($\mathbb{Z}_2$-graded). Suppose that there is a $c > 0$ such that for all $s \in C_c^\infty(X, E)$ supported outside Z, $\|Ds\|_{L^2} \geq c\|s\|_{L^2}$.

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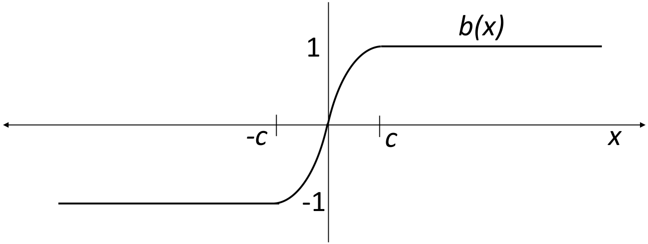
Let $b \in C^\infty(\mathbb{R})$ be odd and increasing, such that $b(x) = \pm 1$ if $|x| \geq c$.



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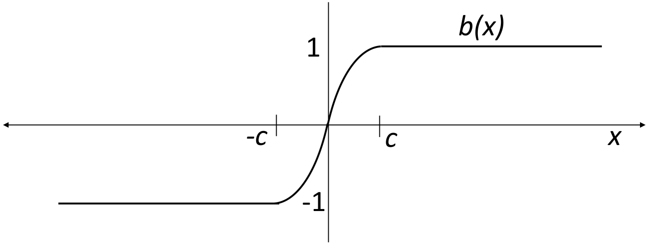

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Theorem (Roe)

1. $b(D) \in D^*(X; Z)$;
2. $S := b(D)^2 - 1 \in C^*(X; Z)$.

Coarse index

The idempotent

$$e := \begin{bmatrix} (S^+)^2 & S^+(1+S^+)b(D)^- \\ S^-b(D)^+ & 1-(S^-)^2 \end{bmatrix} \in C^*(X;Z)^+$$

can be used to construct a coarse index

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and use the geometric representative of the parametrix $R = \phi_1 Q\psi_1 + \phi_2 Q_C\psi_2$ so that

$$S_0 = 1 - RD^+, S_1 = 1 - D^+R \in C^*(X,M)^G$$

and calculate using the heat kernel method.

Higher index

Theorem (Hochs-Wang-W)

Let G acts on a manifold M properly, compactly and isometrically, preserving its boundary N. Let D be a G-invariant Dirac type operator on $M \cup_N N \times [0, \infty)$. Assume the boundary operator D_N has isolated spectrum at 0. Then

$$\mathrm{Ind}_G D = \begin{bmatrix} S_0^2 & S_0(1+S_0)R \\ S_1 D^+ & 1 - S_1^2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in K_0(C^*(X, M)^G).$$

Equivariant η-invariants

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For $g \in G$, the equivariant-η invariant is defined as

$$\eta_g(D_N) := \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr}_g(D_N e^{-t^2 D_N^2}) dt$$

where c is a nonnegative function on N satisfying $\int_G c(gx)dg = 1, \forall x \in N$ and

$$\mathrm{Tr}_g(S) = \int_{G/Z_G(g)} \mathrm{Tr}(hgh^{-1}cS)d(hZ).$$

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Theorem (Hochs-Wang-W)

For proper actions, $\eta_g(D_N)$ is well-defined for G discrete with the conjugacy class (g) having polynomial growth, and for G, g semisimple.

Orbital Integrals

Let $f \in C_c(G)$ and $g \in G$. The orbital integral is defined as

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the orbital integral extends to a continuous trace $\tau_g : \mathcal{A}(G) \to \mathbb{C}$ where $C_c(G) \subset \mathcal{A}(G) \subset C_r^* G$ is closed under holomorphic functional calculus and defines a morphism:

$$\tau_g : K_0(C_r^* G) \cong K_0(\mathcal{A}(G)) \to \mathbb{C}.$$

Main Result

- Let G be a locally compact group acting on a manifold M ($\partial M = N$) properly, compactly and isometrically, preserving N.
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Theorem (Hochs-Wang-W)

When G is either

- *discrete with conjugacy class of g having polynomial growth, or*
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then one has the equivariant APS index formula

$$\tau_g(\mathrm{Ind}_G D) = \int_{M^g} c^g \frac{\hat{A}(M^g)\mathrm{ch}_g(E/S)}{\det(1 - ge^{R|_{N^g}})} - \frac{\eta_g(D_N) + \mathrm{Tr}_g(P_{\ker D_N})}{2}.$$

Corollary

- When $g = e$, for every unimodular group G,

$$L^2\text{- ind}\, D = \int_M c\hat{A}(M)\mathrm{ch}(E/S) - \frac{\eta_{L^2}(D_N) + \mathrm{Tr}(cP_{\ker D_N})}{2}.$$

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- When the action is free and $\ker D_N = \{0\}$ and $g \neq e$,

$$\tau_g(\mathrm{Ind}_G D) = -\frac{\eta_g(D_N)}{2}.$$

Part 3 Mapping Surgery to Analysis

- Γ discrete group free action on $\tilde{M}$, manifold with boundary $\tilde{N}$
- $M := \tilde{M}/\Gamma$ is a compact manifold with boundary $N := \tilde{N}/\Gamma$
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Theorem (Piazza-Schick, Xie-Yu)

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or, equivalently, Yu's exact sequence of localization algebras

$$K_{n+1}(C_L^*\tilde{N}^\Gamma) \to K_{n+1}(C^*\tilde{N}^\Gamma) \to K_n(C_{L,0}^*\tilde{N}^\Gamma) \to K_n(C_L^*\tilde{N}^\Gamma) \to K_n(C^*\tilde{N}^\Gamma)$$

so that all diagrams commute.

Higher ρ-invariant

Higher index of D_M:

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- $\rho_N(h)$ is the obstruction class of $\mathrm{Ind}_\Gamma : K_1(N) \to K_1(C^*_r\Gamma)$.

Delocalized ρ-invariant

Theorem (Xie-Yu)

For $g \neq e$, there exists a map $\omega_g : K_1(C^*_{L,0}\tilde{N}^\Gamma) \to \mathbb{C}$ so that the image of higher ρ-invariant coincides with the Lott's delocalized η-invariant

$$\omega_g(\rho_N(h)) = \eta_g(D_N).$$

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In view of the higher APS index theorem, this is equivalent of saying the commutativity of the diagram (n odd)

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$$\tau_g(\mathrm{Ind}^{APS}_\Gamma D_M) = \eta_g(D_N) = \omega_g(\rho_N(h)).$$

Higher delocalized ρ-numbers

- There is a Chern character map on Higson-Roe exact sequence

$$\begin{array}{ccccccc}
\Omega_{n+1}^{spin}(B\Gamma) & \longrightarrow & R_{n+1}^{spin}(N) & \longrightarrow & Pos_n^{spin}(N) & \longrightarrow & \Omega_n^{spin}(B\Gamma) \\
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K_{n+1}(B\Gamma) & \longrightarrow & K_{n+1}(C_r^*\Gamma) & \longrightarrow & K_{n+1}(D^*\Gamma) & \longrightarrow & K_n(B\Gamma) \\
\downarrow \mathrm{ch}^e & & \downarrow \mathrm{ch} & & \downarrow \mathrm{ch}^{del} & & \downarrow \\
H_\cdot^e(\mathcal{A}\Gamma) & \longrightarrow & H_\cdot(\mathcal{A}\Gamma) & \longrightarrow & H_\cdot^{del}(\mathcal{A}\Gamma) & \longrightarrow & H_\cdot^e(\mathcal{A}\Gamma).
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\downarrow {\scriptstyle \mathrm{ch}^e} & & \downarrow {\scriptstyle \mathrm{ch}} & & \downarrow {\scriptstyle \mathrm{ch}^{del}} & & \downarrow \\
H^e_{\cdot}(\mathcal{A}\Gamma) & \longrightarrow & H_{\cdot}(\mathcal{A}\Gamma) & \longrightarrow & H^{del}_{\cdot}(\mathcal{A}\Gamma) & \longrightarrow & H^e_{\cdot}(\mathcal{A}\Gamma).
\end{array}$$

  so that the diagram commutes.
- Pairing with a delocalized cyclic cocycle of $\mathbb{C}\Gamma$ gives rise to equality between higher delocalized ρ-numbers for N and higher delocalized APS-index (higher delocalized η-invariants).

Higher delocalized ρ-numbers

Theorem (Chen-Wang-Xie-Yu, Piazza-Schick-Zenobi)

For $g \neq e$ and $\phi \in HC^*(\mathbb{C}\Gamma, \langle g \rangle)$, a delocalized cyclic cocycle, its pairing with higher ρ-invariants, denoted $\omega_\phi(\rho_N(h))$, known as delocalized ρ-numbers, are identified as higher delocalized η-invariants

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Remark

Serious analysis on groups is needed in order to have a well-defined pairing. Refer to Chen-Wang-Xie-Yu for precise requirement and estimation.

Thank you!