D-Brane Charges in Wess-Zumino-Witten Models

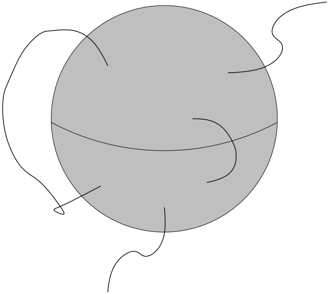
David Ridout

[hep-th/0210302 with P Bouwknegt and P Dawson
hep-th/0312259 with P Bouwknegt
hep-th/0602057 with P Bouwknegt]

October 18, 2010

Strings and Branes

Wess-Zumino-Witten models describe open and closed strings propagating on group manifolds. Dirichlet or D-branes encode the boundary conditions imposed at the ends of open strings, and correspond classically to subspaces.



In M-theory, branes are supposed to be dynamical objects.

Brane Charges

Polchinski used T-duality to argue that D-branes should carry RR-charge in type II string theory on flat space:

$$Q = \int_{\text{brane}} e^F .$$

Here, F is a certain closed 2-form on the brane.

Brane Charges

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Here, F is a certain closed 2-form on the brane.
The extension to curved spaces is due to Minasian and Moore:

$$Q = \int_{\text{brane}} e^{F - \frac{1}{2}c_1(N(\text{brane}))} \frac{\widehat{\mathrm{A}}(T(\text{brane}))}{i^* \sqrt{\widehat{\mathrm{A}}(T(\text{space}))}}.$$

- T and N are the tangent and normal bundles,
- c_1 and $\widehat{\mathrm{A}}$ are the first Chern class and A-roof genus,
- i is the inclusion of the brane into the space.

Charge Groups

Physicists are used to charges which are classified by (real deRham) cohomology groups, *eg*. electric charge. However, Kontsevich and Segal pointed out that the brane charge formula suggests that a K-group is relevant here.

Which K-group?

Charge Groups

Physicists are used to charges which are classified by (real deRham) cohomology groups, *eg*. electric charge. However, Kontsevich and Segal pointed out that the brane charge formula suggests that a K-group is relevant here.

Which K-group?

When there is an NS B-field, Witten proposed that the K-theory should be twisted. When the field strength $H = \mathrm{d}B$ is torsion, he described such a twisted K-theory.
WZW models have non-torsion H. In this case, the appropriate twisted K-theory was proposed by Bouwknegt and Mathai to be an algebraic K-theory constructed by Rosenberg. It reduces to Witten's when H is torsion.

The Plan

- How can we compute the D-brane charge and the charge group?
- Fredenhagen and Schomerus proposed a CFT computation based on the identification of a condensation process for D-branes. They carried out this computation for WZW models on $SU(n)$.
- We extended this to other groups, obtaining predictions for the torsion order of the corresponding twisted K-theories.
- But this is a computation in algebra! This must be reconciled with the geometry that gave rise to the prediction that brane charges were classified by twisted K-theory.

WZW Models as CFTs

Let G be a compact, connected, simply-connected, simple Lie group, *ie*. $\mathsf{G} = \mathsf{SU}(n), \mathsf{Sp}(2n), \mathsf{Spin}(n), \mathsf{G}_2, \mathsf{F}_4, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$. The WZW model on G then defines a family of CFTs, parametrised by the level k, whose symmetry algebra is the corresponding untwisted affine Kac-Moody algebra $\widehat{\mathfrak{g}}_k$.

The consistent D-branes are quantised in bijection with the integrable highest weight modules of $\widehat{\mathfrak{g}}_k$. These branes are conjugacy classes in the group passing through the maximal torus at

$$\exp\left(2\pi\mathrm{i}\frac{\lambda+\rho}{k+\mathsf{h}^{\vee}}\right),$$

where λ is the corresponding dominant integral weight, ρ is the Weyl vector and $\mathsf{h}^{\vee}$ is the dual Coxeter number of $\mathfrak{g}$.

Brane Condensation

A low-energy effective field theory for branes has classical fields A taking values in $\mathfrak{g}$. If we have m coincident branes, a stack, then the components A^a with respect to a basis t_a of $\mathfrak{g}$ are not just real functions, but $m \times m$ matrix-valued functions.

Alekseev, Recknagel and Schomerus wrote down such a field theory action and found its classical equations of motion:

$$\left[A^a, \left[A^a, A^b\right] - f_{abc}A^c\right] = 0.$$

Two obvious solutions:

- $\left[A^a, A^b\right] = 0$ (translation),
- $A^a = \pi(t_a)$ (condensation).

Condensation requires $m = \dim \pi$.

Brane Condensation (cont.)

To interpret, let π_λ be the $\mathfrak{g}$-irrep with highest weight λ and $m = \dim \pi_\lambda$. Then, a stack of m branes labelled by μ can "condense" into the superposition:

$$\dim \pi_\lambda \ \text{brane}_\mu \longrightarrow \bigoplus_\nu {N_{\lambda\mu}}^\nu \ \text{brane}_\nu.$$

Here, $\pi_\lambda \otimes \pi_\mu = \bigoplus_\nu {N_{\lambda\mu}}^\nu \pi_\nu$.

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But this is a classical computation, valid for $k \to \infty$. To quantise, replace $N_{\lambda\mu}{}^\nu$ by the (level k) fusion coefficients $\mathscr{N}_{\lambda\mu}{}^\nu$.
Evidence that this proposal is correct comes from the Kondo model, *à la* Affleck and Ludwig.

Brane Charges

Fredenhagen and Schomerus analysed charges Q_λ conserved under condensation:

$$\dim \pi_\lambda \; Q_\mu = \sum_\nu \mathcal{N}_{\lambda\mu}{}^{\nu} \; Q_\nu .$$

Taking $\mu = 0$ gives $\mathcal{N}_{\lambda\mu}{}^{\nu} = \delta_\lambda^\nu$, hence

$$Q_\lambda = \dim \pi_\lambda .$$

But now we have to satisfy

$$\dim \pi_\lambda \; \dim \pi_\mu = \sum_\nu \mathcal{N}_{\lambda\mu}{}^{\nu} \; \dim \pi_\nu ,$$

which is not true in general. F&S proposed that this holds *modulo* some integer x giving the torsion order of the twisted K-group.

Example: $SU(2)$

Fusion defines a commutative associative operation on the integrable highest weight modules of $\widehat{\mathfrak{g}}_k$. The fusion ring may then be described as a quotient of the representation ring of $\mathfrak{g}$.

For $\widehat{\mathfrak{sl}}(2)_k$, the fusion ring is the quotient by the ideal generated by $\pi_{(k+1)\Lambda}$ (Λ is the fundamental weight). Thus,

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This correctly gives the torsion order of the twisted K-theory

$${}^{H}\mathrm{K}^{*}(\mathrm{SU}(2)) \cong \mathbb{Z}_{k+2},$$

when H is a closed 3-form represented in $\mathrm{H}^3(\mathrm{SU}(2);\mathbb{Z}) \cong \mathbb{Z}$ by $k+2$.

Example: SU(3)

For $\widehat{\mathfrak{sl}}(3)_k$, the fusion ring is the quotient by the ideal generated by $\pi_{(k+1)\Lambda_1}$ and $\pi_{(k+2)\Lambda_1}$, hence

$$x = \gcd\left\{\dim \pi_{(k+1)\Lambda_1}, \dim \pi_{(k+2)\Lambda_1}\right\}$$
$$= \gcd\left\{\binom{k+3}{2}, \binom{k+4}{2}\right\} = \frac{k+3}{\gcd\{k+3, 2\}}.$$

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The twisted K-theory was not known at the time, only that its torsion order divided $k+3$. Maldacena, Moore and Seiberg subsequently gave a physical computation, obtaining

$$^{H}\mathrm{K}^*(\mathrm{SU}(3)) \cong \mathbb{Z}_x \oplus \mathbb{Z}_x,$$

when H is represented in $\mathrm{H}^3(\mathrm{SU}(3);\mathbb{Z}) \cong \mathbb{Z}$ by $k+3$.

Example: $SU(n)$

Using induction and a modified Littlewood-Richardson rule for fusion products, Fredenhagen and Schomerus proved:

Theorem (Fredenhagen–Schomerus)

For $\widehat{\mathfrak{sl}}(n)_k$, the maximal possible torsion order for the D-brane charge group is

$$x = \frac{k+n}{\gcd\{k+n, \mathrm{lcm}\{1,2,\ldots,n-1\}\}}.$$

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Maldacena, Moore and Seiberg reproduced this result by imposing invariance under affine outer automorphisms and announced that Hopkins had shown that

$$^{H}\mathrm{K}^{*}(\mathrm{SU}(n)) \cong \mathbb{Z}_x \otimes \bigwedge{}^{*}[w_5, w_7, \ldots, w_{2n-1}] \sim \mathbb{Z}_x^{\oplus 2^{n-2}},$$

when H is represented in $\mathrm{H}^3(\mathrm{SU}(n);\mathbb{Z}) \cong \mathbb{Z}$ by $k+n$.

General Charge Groups

We simplified the proof of this theorem using better generators for the fusion ideal and generalised it to $\mathsf{Sp}(2n)$:

Theorem

For $\widehat{\mathfrak{sp}}(2n)_k$, the maximal possible torsion order for the D-brane charge group is

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Moreover, we conjectured (based on numerics) that in all other cases,

$$x = \frac{k+\mathrm{h}^{\vee}}{\gcd\{k+\mathrm{h}^{\vee}, \mathrm{lcm}\{1,2,\ldots,\mathrm{h}-1\}\}}.$$

Twisted K-Theory Computations

Shortly thereafter, Braun showed that when the fusion ideal for $\widehat{\mathfrak{g}}_k$ is generated by $r = \operatorname{rank} \mathfrak{g}$ elements, then

$$ {}^{H}\mathrm{K}^{*}(\mathrm{G}) \cong \mathbb{Z}_x^{\oplus 2^{r-1}}, $$

and the determination of x proceeds *à la* Fredenhagen and Schomerus.

Douglas then showed directly (following Hopkins) that

$$ {}^{H}\mathrm{K}^{*}(\mathrm{G}) \cong \mathbb{Z}_x \otimes \bigwedge^{*} [w_1, w_2, \ldots, w_{r-1}], $$

and he computed x in all cases except $\mathrm{G} = \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$. He then went on to compute generators for the fusion ideal, using Freed-Hopkins-Teleman, in every case!

Where do we stand?

We have a physical computation of brane charges and the torsion orders of their charge groups. We also have the twisted K-theories. Agreement has been reached!

But, the physics has largely ignored the geometric nature of the problem. Instead, the computations rest upon a conjectured quantisation of a classical low-energy effective field theory.

We have seen that physicists had earlier predicted a geometric form for the brane charge. For (the nicest) branes on our Lie groups, we should therefore have

$$\int_{C_\lambda} e^{F_\lambda} \mathrm{Td}\left(T\left(C_\lambda\right)\right) \stackrel{?}{=} \dim \pi_\lambda ,$$

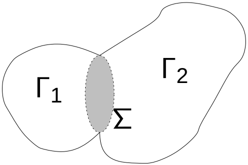
where C_λ is the conjugacy class (brane) through $\exp\left(2\pi \mathrm{i}\left(\lambda+\rho\right)/\left(k+\mathsf{h}^\vee\right)\right)$, F_λ is a certain closed 2-form on the brane, and π_λ is the irrep with highest weight λ.

WZW Models — Closed Strings

The closed string WZW action consists of a standard kinetic term and a Wess-Zumino term:

$$S_{\mathrm{WZ}} = 2\pi \mathrm{i} \int_{\tilde{g}(\Gamma)} H, \qquad H = \frac{k}{24\pi^2} \kappa(\theta \wedge \mathrm{d}\theta).$$

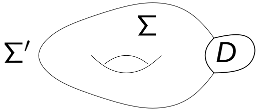
$\tilde{g} : \Gamma \to \mathsf{G}$ extends the string map $g : \Sigma \to \mathsf{G}$ in the sense that $\partial\Gamma = \Sigma$ (note $\mathrm{H}_2(\mathsf{G};\mathbb{Z}) = 0$). As H is cohomologically non-trivial, $[H] = k$ in $\mathrm{H}^3(\mathsf{G};\mathbb{Z})$, the action depends upon the choice of Γ.



However, the Feynman amplitudes e^{-S} are well-defined.

WZW Models — Open Strings

For open strings, the worldsheet Σ has non-trivial boundary, $\partial\Sigma = S^1$. There are no Γ with $\partial\Gamma = \Sigma$, so extend Σ to $\Sigma' = \Sigma + D$ and $g : \Sigma \to \mathrm{G}$ to $g' : \Sigma' \to \mathrm{G}$ (note $\mathrm{H}_2(\mathrm{G};\mathbb{Z}) = 0$).



We can now define the WZ term by extending g' to $\tilde{g}' : \Gamma' \to \mathrm{G}$, where $\partial\Gamma' = \Sigma'$. This is then modified as follows:

$$S_{\mathrm{WZ}} = 2\pi\mathrm{i}\left[\int_{\tilde{g}'(\Gamma)} H - \int_{g'(D)} \omega\right].$$

Here, ω is a 2-form on (a tubular neighbourhood of) $g'(D)$ where $\mathrm{d}\omega = H$. It "cancels" the effect of patching Σ with D.

Boundary Conditions

The precise form of ω depends upon the boundary conditions chosen for the open string endpoints. We take

$$\partial g = -\operatorname{Ad}(g)\,\bar{\partial} g,$$

which implies that the D-branes are conjugacy classes C and that ω is the 2-form on C given by

$$g^{*}\omega = \frac{-k}{16\pi^2}\kappa\left(g^{-1}\mathrm{d}g \wedge \frac{\mathrm{id}+\operatorname{Ad}(g)}{\mathrm{id}-\operatorname{Ad}(g)}\,g^{-1}\mathrm{d}g\right).$$

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Now, g extends to g' since $\mathrm{H}_1(C;\mathbb{Z}) = 0$ and g' extends to $\tilde{g}'$ since $\mathrm{H}_2(\mathrm{G};\mathbb{Z}) = 0$. Equivalently, Σ needs to be a boundary *modulo* C, which follows from $\mathrm{H}_2(\mathrm{G},C;\mathbb{Z}) = 0$.

The amplitudes are well-defined if $(H,\omega) \in \mathrm{H}^3(\mathrm{G},C;\mathbb{Z})$. This quantises the D-branes: $C \to C_\lambda$, $\omega \to \omega_\lambda$.

The $\mathrm{U}(1)$-flux F_λ

The quantised branes are the conjugacy classes C_λ passing through $\exp(2\pi\mathrm{i}(\lambda+\rho)/(k+\mathrm{h}^\vee))$, hence each is homeomorphic to G/T. H is exact on each brane, so physicists write $H=\mathrm{d}B$ and "define" a closed 2-form by

$$F_\lambda = B - \omega_\lambda\,.$$

What they mean is take the closed 2-form whose periods are

$$\int_S F_\lambda = \int_S (B-\omega_\lambda) = \int_Z H - \int_S \omega_\lambda \qquad (\partial Z = S),$$

from which we get $F_\lambda \in \mathrm{H}^2(C_\lambda;\mathbb{Z})$. Note that this is still ambiguous up to the periods of H.

Example: $\mathrm{SU}(2)$

We need to evaluate the following integral:

$$Q_\lambda = \int_{C_\lambda} e^{F_\lambda} \mathrm{Td}(T(C_\lambda)) = \int_{S^2} \left(F_\lambda + \frac{1}{2} c_1 \left(T(S^2) \right) \right).$$

Parametrising $\mathrm{SU}(2)$ explicitly to get H and ω_λ, we find that $\int F_\lambda = (\lambda, \alpha)$. Thus,

$$Q_\lambda = (\lambda, \alpha) + 1 = \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)} = \dim \pi_\lambda .$$

This computation is due to Bachas–Douglas–Schweigert; Stanciu; Alekseev–Schomerus; ...

Moreover, if H is represented by $k+2$ in cohomology, the period of F_λ is ambiguous up to factors of $k+2$, hence the charge is only well-defined *modulo* $k+2$.

Required Knowledge

Our branes are homeomorphic to G/T, so we'll need to know about the cohomology rings of this space.

Fact (Bott)

The (co)homology ring of G/T *is torsion-free and concentrated in even degree. It has a natural basis in bijection with the Weyl group* W *of* G *such that the degree of each basis element is twice the length of the corresponding Weyl transformation.*

Now, we have a natural sequence of isomorphisms:

$$H_2(G/T;\mathbb{Z}) \cong \pi_2(G/T) \cong \pi_1(T) \cong \ker(\exp\colon \mathfrak{t} \to T) = Q^\vee.$$

The second integral homology of our branes may be identified with the coroot lattice of G.

Required Knowledge (cont.)

Likewise, the second integral cohomology of our branes may be identified with the weight lattice of G:

$$\mathrm{H}^2(\mathrm{G}/\mathrm{T};\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{Q}^\vee,\mathbb{Z}) = \mathrm{P}.$$

With this formalism, one can sharpen Bott's result to give the ring structure, at least over the rationals.

Fact (Borel)

The rational cohomology ring of G/T *has the form*

$$\mathrm{H}^*(\mathrm{G}/\mathrm{T};\mathbb{Q}) \cong \frac{\mathbb{Q}[\Lambda_1,\Lambda_2,\ldots,\Lambda_r]}{I_+},$$

where r *is the rank of* G*,* Λ_i *denotes the fundamental weights, and* I_+ *is the ideal of* W*-invariant polynomials of positive degree.*

Required Knowledge (cont.)

The splitting principle states that as far as characteristic classes are concerned, any vector bundle may be replaced by an appropriate sum of line bundles. If E splits as $\bigoplus_{i=1}^{n} L_i$, then the Todd class of E is

$$\mathrm{Td}(E) = \prod_{i=1}^{n} \frac{\mathrm{c}_1(L_i)}{1 - e^{-\mathrm{c}_1(L_i)}}.$$

The tangent bundle of G/T is a complex vector bundle whose rank is the number $|\Delta_+|$ of positive roots of G. This suggests:

Fact (Borel–Hirzebruch)

The first Chern classes of the line bundles associated with $T(\mathrm{G}/\mathrm{T})$ under the splitting principle are, in Borel's formalism, precisely the positive roots of G.

Computing the Charge

Note first that in Borel's formalism, $F_\lambda \in \mathrm{H}^2(\mathrm{G}/\mathrm{T};\mathbb{Z})$ is represented by $\lambda \in \mathrm{P}$. We therefore compute

$$Q_\lambda = \int_{\mathrm{G}/\mathrm{T}} e^{F_\lambda} \mathrm{Td}(T(\mathrm{G}/\mathrm{T})) = \int_{\mathrm{G}/\mathrm{T}} e^\lambda \prod_{\alpha\in\Delta_+} \frac{\alpha}{1-e^{-\alpha}}$$
$$= \int_{\mathrm{G}/\mathrm{T}} \frac{e^\lambda}{\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})} \prod_{\alpha\in\Delta_+} \alpha. \qquad (1)$$

Now, the product of the roots is a volume form:

$$\int_{\mathrm{G}/\mathrm{T}} \prod_{\alpha\in\Delta_+} \alpha = \int_{\mathrm{G}/\mathrm{T}} c_{|\Delta_+|}(T(\mathrm{G}/\mathrm{T})) = \chi(\mathrm{G}/\mathrm{T}) = |\mathrm{W}|. \qquad (2)$$

We therefore need to extract the degree zero terms in the rest of the integrand of (1).

Computing the Charge (cont.)

This is hard! The factor

$$\frac{e^{\lambda}}{\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})}$$

has an order $|\Delta_+|$ pole at the origin. We recognise it as the character of the (infinite-dimensional!) Verma module of highest weight λ.

To obtain a better character, note that the volume form is anti-invariant under W:

$$w\left(\prod_{\alpha\in\Delta_+}\alpha\right)=(-1)^{\ell(w)}\prod_{\alpha\in\Delta_+}\alpha=\det w\prod_{\alpha\in\Delta_+}\alpha.$$

It follows that antisymmetrising (1) will not change the value of the integral.

Computing the Charge (cont.)

But, antisymmetrisation gives the Weyl character formula for the irreducible module of highest weight λ:

$$
\begin{aligned}
Q_\lambda &= \frac{1}{|\mathsf{W}|} \sum_{w \in \mathsf{W}} \det w \int_{\mathsf{G}/\mathsf{T}} w \left(e^{\lambda} \prod_{\alpha \in \Delta_+} \frac{\alpha}{1 - e^{-\alpha}} \right) \\
&= \frac{1}{|\mathsf{W}|} \int_{\mathsf{G}/\mathsf{T}} \sum_{w \in \mathsf{W}} \det w \; w \left(\frac{e^{\lambda+\rho}}{\prod_{\alpha \in \Delta_+} \left(e^{\alpha/2} - e^{-\alpha/2} \right)} \prod_{\alpha \in \Delta_+} \alpha \right) \\
&= \frac{1}{|\mathsf{W}|} \int_{\mathsf{G}/\mathsf{T}} \frac{\sum_{w \in \mathsf{W}} \det w \; e^{w(\lambda+\rho)}}{\prod_{\alpha \in \Delta_+} \left(e^{\alpha/2} - e^{-\alpha/2} \right)} \prod_{\alpha \in \Delta_+} \alpha .
\end{aligned}
$$

Extracting the degree zero term and using (2) then gives

$$
Q_\lambda = \frac{\dim \pi_\lambda}{|\mathsf{W}|} \int_{\mathsf{G}/\mathsf{T}} \prod_{\alpha \in \Delta_+} \alpha = \dim \pi_\lambda .
$$

Discussion

- Borel's characterisation of the cohomology ring gives

$$Q_\lambda = \int_{\mathrm{G/T}} e^{\lambda+\rho}.$$

- The conjugacy class C_λ only determines λ up to the shifted action of the affine Weyl group. The charge must therefore be invariant under this action. When $\mathrm{G} \neq \mathrm{Sp}(2n)$, this is equivalent the dimension constraints of Fredenhagen and Schomerus!
- The periods of the 2-form F_λ are only determined up to periods of the 3-form H, hence the charge group must be invariant under this ambiguity. This only affects the charge group torsion when $\mathrm{G} = \mathrm{Sp}(2n)$. Curiously, certain symplectic groups now have torsion order half that of the K-theory!?!