T-duality and the Bulk-Boundary Correspondence

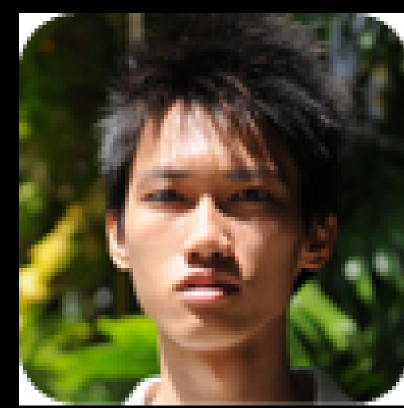
Keith Hannabuss

IGA/AMSI Workshop, Adelaide 2016:
Topological matter, strings,
K-theory, and related areas.

28 September 2016

Joint work with

Mathai and Guo Chuan

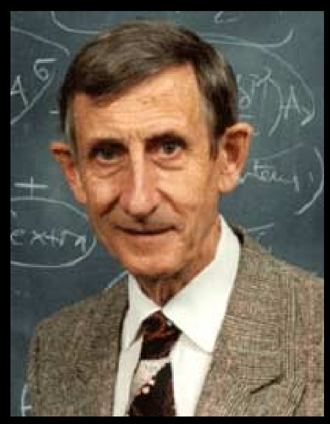


Mathai+Thiang `arXiv:1503.01206, 1505.05250, 1506.04492`;

H+Mathai+Thiang `arXiv:1510.04785, 1603.00116`.

Outline

1. Historical Introduction;
2. Topological Insulators - Bulk and boundary;
3. C^*-algebras and noncommutative geometry;
4. The physical picture of bulk and boundary;
5. The geometrical picture of bulk and boundary;
6. The geometrical and physical pictures as T-duals;
7. Possible applications of flux.

1932, 1950s	1962
	
E.P. Wigner	F. Dyson
1932: Unitary-Antiunitary Theorem	9-fold way
1951-5: 3-fold way	$\mathbb{RR}, \mathbb{RC}, \mathbb{RH}$
$\mathbb{R}, \mathbb{C}, \mathbb{H}$	$\mathbb{CR}, \mathbb{CC}, \mathbb{CH}$
	$\mathbb{HR}, \mathbb{HC}, \mathbb{HH}$
	Symmetric spaces

1996

M. Zirnbauer

+Altland

10-fold way

Chirality

2005

C.L. Kane

+Fu+Mele

Topological order

$\mathbb{Z}_2$ phases

2009

A. Kitaev

Periodic Table

K-theory

The 10-fold way - Altland and Zirnbauer

Cartan	T	C	P	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Kitaev: Table displays Bott periodicities of K/KR-theory

Cartan	T	C	P	1	2	3	4	5	6	7	8
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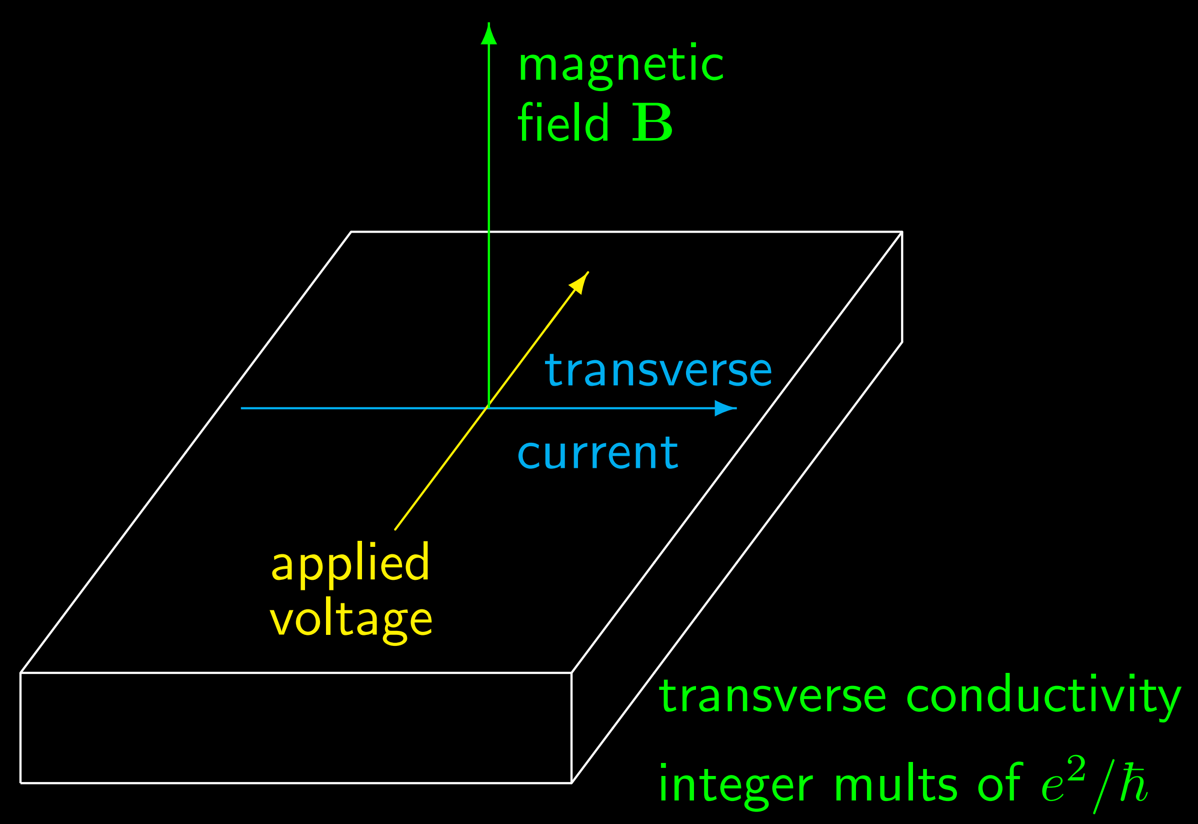
Integer Quantum Hall Effect (Klitzing et al., 1980)

D. Thouless et al (1982): Quantum Hall conductivity as $c_1(\mathbb{T}^2) = c_1(\mathrm{BZ})$.



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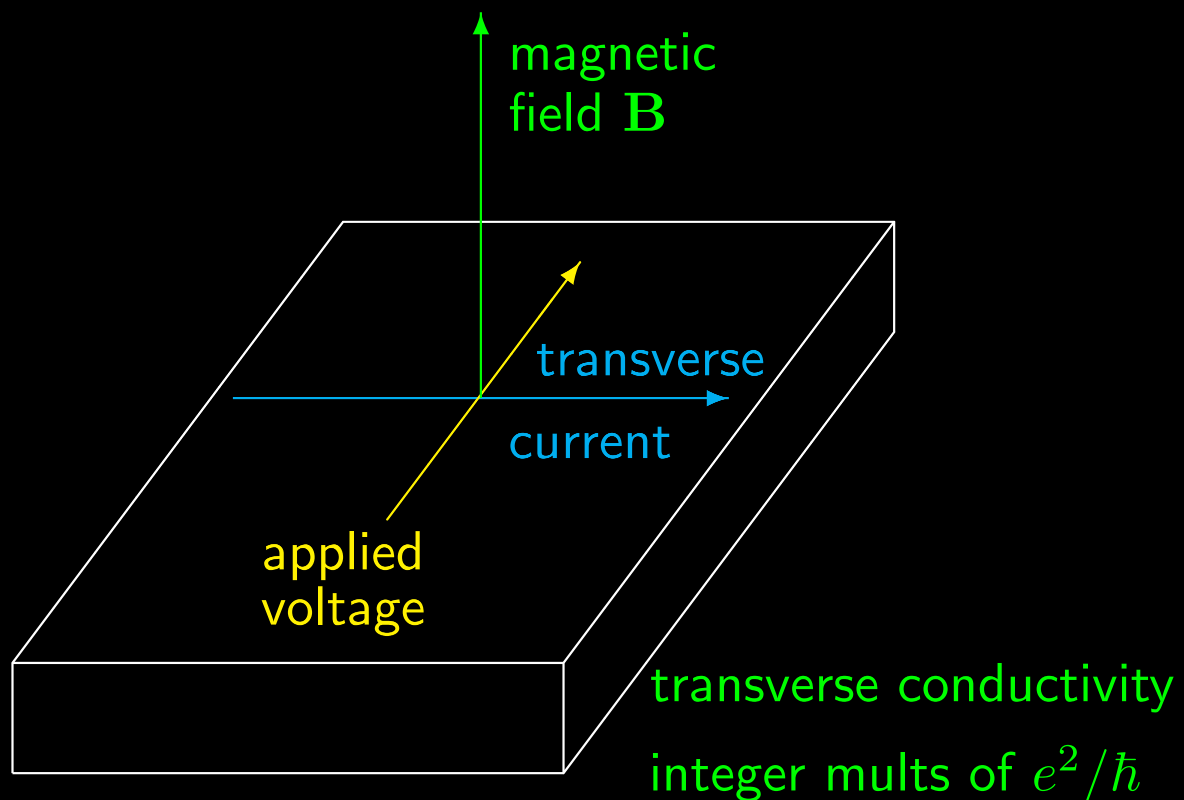
J. Bellissard 1985

Noncommutative Geometry



C*-algebras

The observables are modelled by a C*-algebra, which can be defined as a *-subalgebra of the bounded operators $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ closed with respect to the norm metric:

$$\|A\| = \sup\{\|A\psi\| : \|\psi\| = 1\} < \infty$$

C*-algebra Examples

*-subalgebra of the bounded operators $\mathcal{B}(\mathcal{H})$:

EXAMPLE: $\mathcal{A} = C_0(P)$ pointwise multiplication on $\mathcal{H} = L^2(P)$.

C*-ALGEBRAS

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EXAMPLE: Compact operators: $\mathcal{K}(\mathcal{H})$ closure of finite rank operators on $\mathcal{H}$.

EXAMPLE: The Toeplitz algebra $\mathcal{T}$ on $\mathcal{H} = \ell^2(\mathbb{N})$ is generated by the shift operator S such that $(Sf)(0) = 0$, and $(Sf)(k) = f(k-1)$, for $k = 1, 2, \ldots$

Morita Equivalence.

Algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are Morita equivalent if there is a natural equivalence between the categories of $\mathcal{A}_1$-modules and $\mathcal{A}_2$-modules.

Example: The compact operators $\mathcal{K}(\mathcal{H})$ have only one irreducible representation (the obvious one defined on $\mathcal{H}$), so they are all Morita equivalent to each other, and, in particular, to $\mathcal{K}(\mathbb{C}) \cong \mathbb{C}$.

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Example: The quantum mechanical commutation relations have a unique irreducible representation (Stone–von Neumann Theorem), so they are also Morita equivalent to $\mathcal{K}(\mathcal{H})$, and to $\mathbb{C}$.

Algebra-Topology dictionary

Geometry	$\longleftrightarrow$	Algebra
	Gel'fand-Naimark:	
loc. cpt Hausdorff space P	$\longrightarrow$	comm. C*-algebra $C_0(P)$
spectrum of $\mathcal{A}$ (irreps)	$\longleftarrow$	comm. C*-algebra $\mathcal{A}$
	Serre-Swan:	
vector bundle over P	$\longleftrightarrow$	finite rank projective $C_0(P)$-module
$K^*(P)$	$\longleftrightarrow$	$K_*(C_0(P)) \sim$ projections

K-theory of C*-algebras.

For any C*-algebra $\mathcal{A}$ we may define $K_0(\mathcal{A})$ to be the equivalence classes of projections in $M_\infty(\mathcal{A})$, the matrices of arbitrary size with entries in $\mathcal{A}$,

where homotopy equivalence, unitary equivalence, and von Neumann equivalence ($p = u^*u \sim q = uu^*$), amongst others, all give the same K-theory.

One can also define $K_1(\mathcal{A}) = K_0\left(C_0(\mathbb{R}, \mathcal{A})\right)$.

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Morita equivalent algebras have the same K-theory.

Bulk and boundary: the Toeplitz algebra

The Toeplitz algebra $\mathcal{T}$ generated by the shift S on $\ell^2(\mathbb{N})$, plays an important role for the unit disc $\mathbb{D}$ with boundary unit circle S^1.

The Hardy space $H^2(S^1)$, is the subspace of functions in $L^2(S^1)$ which extend holomorphically to $\mathbb{D}$; there is an associated (positive energy) projection $P : L^2(S^1) \to H^2(S^1)$.

The pointwise multiplication action of $C_0(S^1)$ on $L^2(S^1) \cong \ell^2(\mathbb{Z})$ can be restricted to $H^2(S^1) \cong \ell^2(\mathbb{N})$ and gives an action of the Toeplitz algebra:

$$\mathcal{T} \cong P\, C_0(S^1)\, P.$$

The algebraic bulk-boundary correspondence

J. Kellendonk, T. Richter and H. Schulz-Baldes (2002)

$$\text{boundary} \longrightarrow \text{"glue"} \longrightarrow \text{bulk}$$

or

$$0 \longrightarrow \mathcal{K} \otimes \widehat{\mathcal{E}} \longrightarrow \mathcal{T}(\widehat{\mathcal{E}}) \longrightarrow \widehat{\mathcal{B}} \longrightarrow 0,$$

where $\mathcal{T}(\widehat{\mathcal{E}}) \leq \mathcal{T} \otimes \widehat{\mathcal{E}}$ is a Toeplitz algebra.

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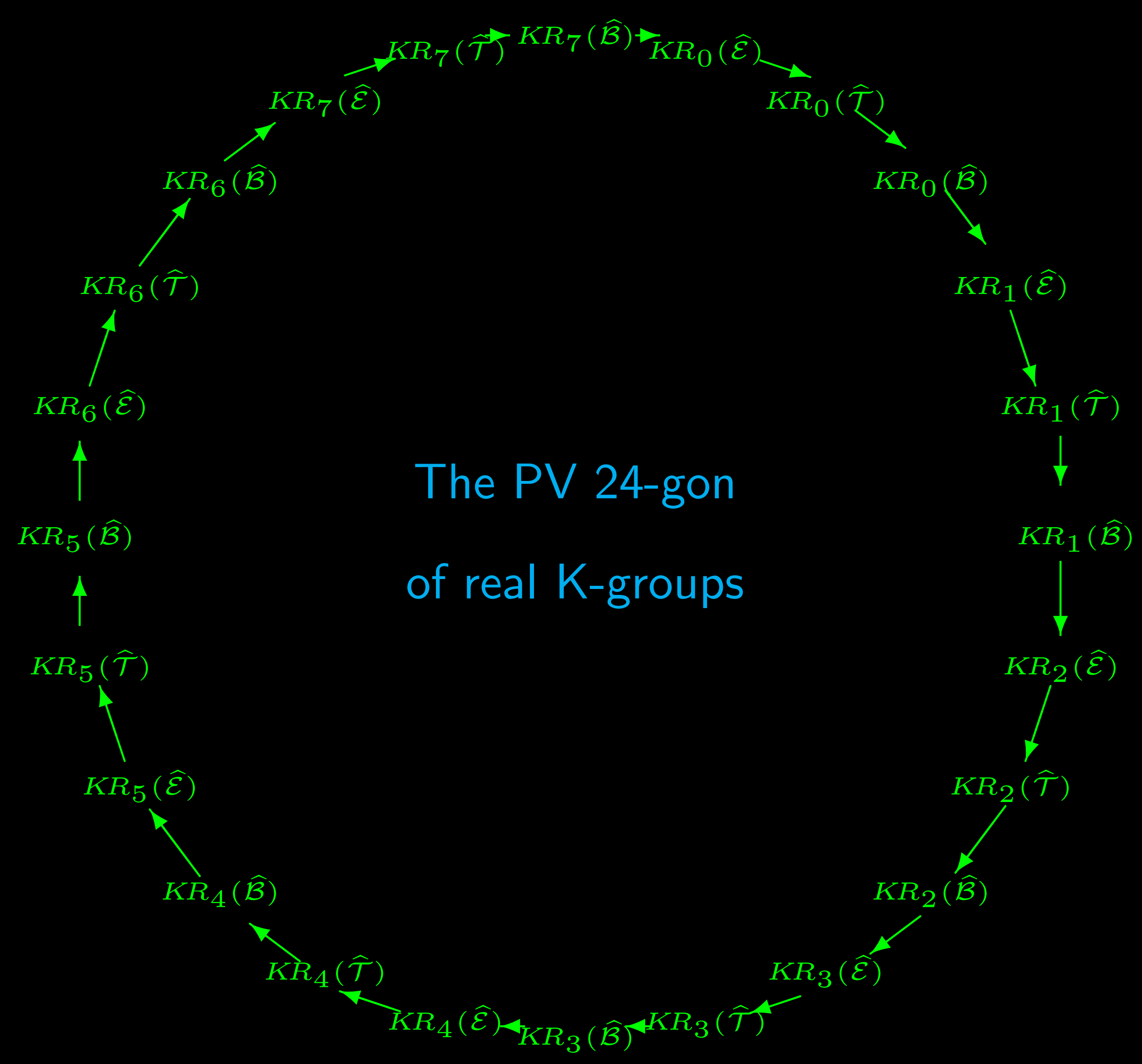
There is a Pimsner–Voiculescu (PV) exact hexagon of the algebraic K-groups:

$$\begin{array}{ccccc} K_0(\widehat{\mathcal{E}}) & \longrightarrow & K_0(\mathcal{T}) & \longrightarrow & K_0(\widehat{\mathcal{B}}) \\ \uparrow & & & & \downarrow \\ K_1(\widehat{\mathcal{B}}) & \longleftarrow & K_1(\mathcal{T}) & \longleftarrow & K_1(\widehat{\mathcal{E}}). \end{array}$$

It can be shown that $K_j(\mathcal{T}(\widehat{\mathcal{E}})) \cong K_j(\widehat{\mathcal{E}})$.

$$\begin{array}{ccccc} K_0(\widehat{\mathcal{E}}) & \longrightarrow & K_0(\mathcal{E}) & \longrightarrow & K_0(\widehat{\mathcal{B}}) \\ \uparrow & & & & \downarrow \\ K_1(\widehat{\mathcal{B}}) & \longleftarrow & K_1(\mathcal{E}) & \longleftarrow & K_1(\widehat{\mathcal{E}}). \end{array}$$

The vertical arows in the Pimsner–Voiculescu (PV) are not so easy to handle


The PV 24-gon
of real K-groups
$KR_7(\widehat{\mathcal{B}})$
$KR_0(\widehat{\mathcal{E}})$
$KR_0(\widehat{\mathcal{T}})$
$KR_0(\widehat{\mathcal{B}})$
$KR_1(\widehat{\mathcal{E}})$
$KR_1(\widehat{\mathcal{T}})$
$KR_1(\widehat{\mathcal{B}})$
$KR_2(\widehat{\mathcal{E}})$
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$KR_6(\widehat{\mathcal{B}})$
$KR_7(\widehat{\mathcal{E}})$
$KR_7(\widehat{\mathcal{T}})$

Periodic potentials

Bloch–Floquet Theory Stationary solutions of Schrödinger's equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

with periodic potential $V(\mathbf{x}+\mathbf{a}_j) = V(\mathbf{x})$, for $j = 1, 2, 3$ can be written as

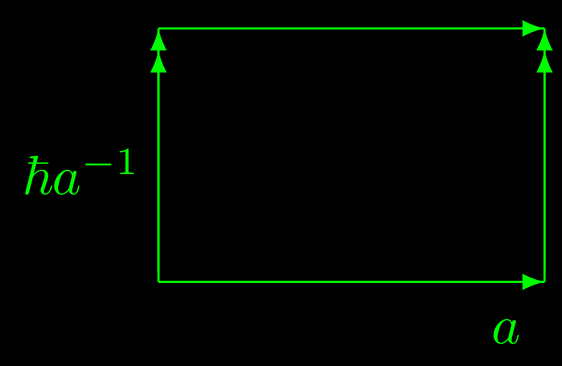
$$\psi(\mathbf{x}) = e^{i\mathbf{k}.\mathbf{x}/\hbar}u_{\mathbf{k}}(\mathbf{x})$$

where $u_{\mathbf{k}}(\mathbf{x})$ ihas the same periodicity as V, and $\mathbf{k}$ has periodicity with respect to integral combinations of $\mathbf{a}'_1 = \hbar\mathbf{a}_2 \times \mathbf{a}_3/[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$, $\mathbf{a}'_2$, $\mathbf{a}'_3$, and the energy comes in bands $E_n(\mathbf{k})$, for $n \in \mathbb{N}$.

The Lattice Representation of the CCR

As well as the Schrödinger representation of the canonical commutation relations on $L^2(\mathbb{R}^d)$, and the Fock–Bargmann–Segal representation on square-integrable holomorphic functions on $\mathbb{C}^d$, there is Cartier's lattice representation on L^2 sections of a line bundle over $\mathbb{T}^{2d}$, induced from a lattice in phase space, which for $d = 3$ can be taken to be generated by $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3\}$.

For $d = 1$ on $\mathbb{T}^2$ with periods $\{a, \hbar/a\}$:



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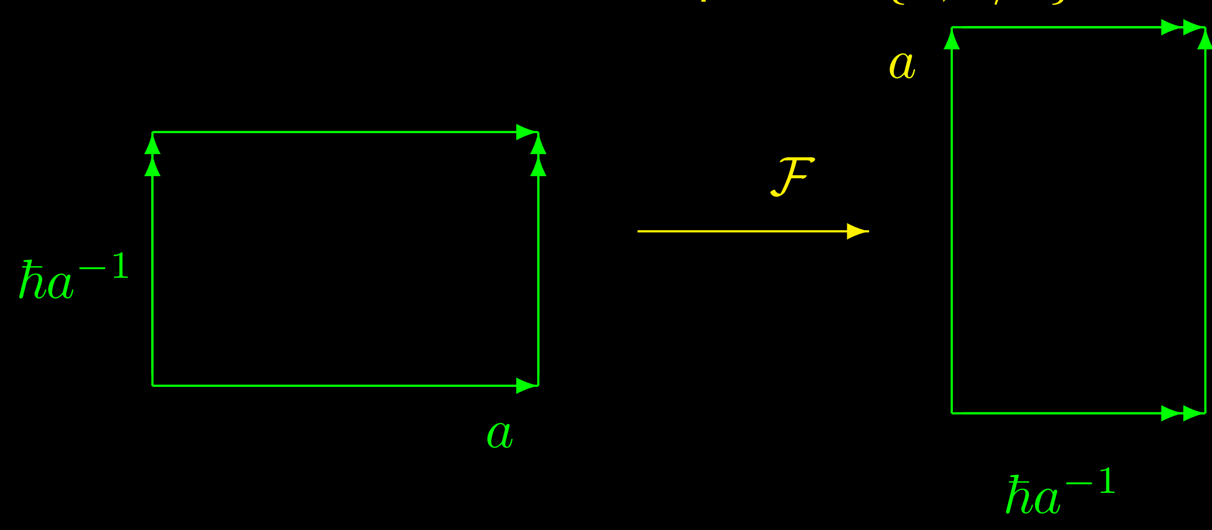
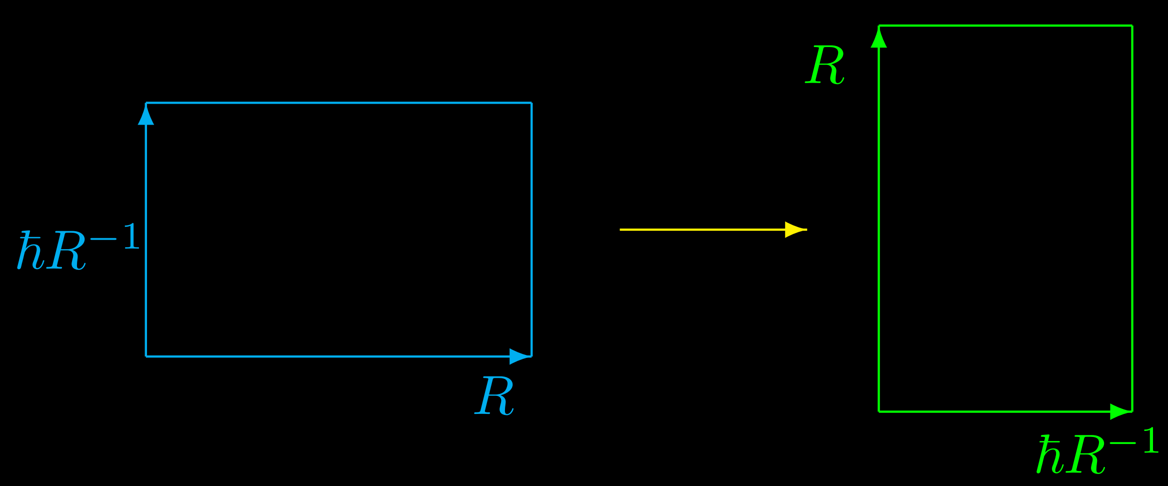


pointwise multiplication $\longrightarrow$ convolution multiplication
pointwise evaluation $\longrightarrow$ integration

T-DUALITY: $R \leftrightarrow \hbar R^{-1}$ PRESERVES THE PHYSICS



STRING THEORY (BUSCHER, HULL AND TOWNSEND, ...)

T-duality: Momentum and winding number interchange

Added ingredient: flux $H \in H^3(P)$

$\alpha : V \to \mathrm{Aut}(\mathcal{A})$.

Crossed product $\widehat{\mathcal{A}} = \mathcal{A} \rtimes_\alpha V = C_0(V, \mathcal{A})$ has α-twisted convolution

$$\begin{aligned}(f * g)(v) &= \int_V f(u)\alpha_u[g(v-u)]\,du,\\ f^*(v) &= \alpha_v[f(-v)]^*\end{aligned}$$

The T-dual of $\mathcal{A}$ with $\alpha : V \to \mathrm{Aut}(\mathcal{A})$ is the crossed product

$$\widehat{\mathcal{A}} = \mathcal{A} \rtimes_\alpha V.$$

A lattice L in V acts trivially on the spectrum, so it looks more like the action of the torus V/L.

TAKAI–TAKESAKI T-DUALITY

For V abelian: the Pontryagin dual $\widehat{V} = \mathrm{Hom}(V, \mathbb{T}) \subset C_0(V, \mathbb{T})$ acts by multiplication on the T-dual $\mathcal{A} \rtimes_\alpha V$.

(TAKAI-TAKESAKI DUALITY) $\widehat{\mathcal{A}} \rtimes \widehat{V} \cong \mathcal{A} \otimes \mathcal{K}(L^2(V)) \sim_M \mathcal{A}$.

The T-dual of $\widehat{\mathcal{A}}$ is Morita equivalent to $\mathcal{A}$.

In the previous discussion we may take

$$\widehat{\mathcal{B}} := \widehat{\mathcal{E}} \rtimes_{\alpha'} \mathbb{Z}.$$

The geometrical picture

Schematically we expect

$$\text{interior} \longrightarrow \text{bulk} \longrightarrow \text{boundary}$$

Algebras:

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{B} \longrightarrow \mathcal{E} \longrightarrow 0.$$

Induced Algebras: pointwise multiplication

Let α be a homomorphism from $\mathbb{Z}$ to the automorphisms $\mathrm{Aut}(\mathcal{E})$.

The induced algebra consists of $\mathcal{E}$-valued functions on $\mathbb{R}$, with a periodicity condition:

$$\mathrm{ind}_{\mathbb{Z}}^{\mathbb{R}}(\mathcal{E},\alpha) = \{f \in C_0(\mathbb{R},\mathcal{E}) : f(x-n) = \alpha(n)[f(x)]\}$$

and with the pointwise product.

Set $$\mathcal{B} = \mathrm{ind}_{\mathbb{Z}}^{\mathbb{R}}(\mathcal{E},\alpha)$$

Also set $$\mathcal{I} = C_0((0,1),\mathcal{E}) \cong C_0(\mathbb{R},\mathcal{E}).$$

Functions in $\mathcal{I} = C_0((0,1),\mathcal{E})$ extend "periodically" to give functions in $\mathcal{B}$.

Functions in $\mathcal{B}$ can be evaluated at 0 to give functions in $\mathcal{E}$, and $\mathcal{I}$ is the kernel of this map.

The maps $\mathcal{I} \to \mathcal{B}$ and $\mathcal{B} \to \mathcal{E}$ give the expected exact sequence

$$0 \longrightarrow C_0((0,1),\mathcal{E}) \longrightarrow \mathrm{ind}_{\mathbb{Z}}^{\mathbb{R}}(\mathcal{E},\alpha) \longrightarrow \mathcal{E} \longrightarrow 0.$$

or

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{B} \longrightarrow \mathcal{E} \longrightarrow 0.$$

Again there is a PV exact hexagon of the algebraic K-groups:

$$\begin{array}{ccccc} K_0(\mathcal{I}) & \longrightarrow & K_0(\mathcal{B}) & \longrightarrow & K_0(\mathcal{E}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{E}) & \longleftarrow & K_1(\mathcal{B}) & \longleftarrow & K_1(\mathcal{I}). \end{array}$$

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When the algebras are related by T-duality,
the geometric and physical PV hexagons are the same up to degree shifts!

The geometrical picture rewritten

The interior algebra $\mathcal{I} = C_0(\mathbb{R}, \mathcal{E})$ is the suspension of $\mathcal{E}$, so that $K_j(\mathcal{I}) = K_{j-1}(\mathcal{E})$.

So the PV hexagon:

$$\begin{array}{ccccc} K_0(\mathcal{I}) & \longrightarrow & K_0(\mathcal{B}) & \longrightarrow & K_0(\mathcal{E}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{E}) & \longleftarrow & K_1(\mathcal{B}) & \longleftarrow & K_1(\mathcal{I}). \end{array}$$

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So the PV hexagon becomes:

$$\begin{array}{ccccc} K_1(\mathcal{E}) & \longrightarrow & K_0(\mathcal{B}) & \longrightarrow & K_0(\mathcal{E}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{E}) & \longleftarrow & K_1(\mathcal{B}) & \longleftarrow & K_0(\mathcal{E}). \end{array}$$

The geometrical picture rewritten

The interior algebra $\mathcal{I} = C_0(\mathbb{R}, \mathcal{E})$ is the suspension of $\mathcal{E}$, so that $K_j(\mathcal{I}) = K_{j-1}(\mathcal{E})$.

So the PV hexagon rotates to:

$$\begin{array}{ccccc} K_1(\mathcal{E}) & \longrightarrow & K_1(\mathcal{E}) & \longrightarrow & K_0(\mathcal{B}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{B}) & \longleftarrow & K_0(\mathcal{E}) & \longleftarrow & K_0(\mathcal{E}). \end{array}$$

Can we reconstruct the physical picture from the geometrical picture?

Given $\mathcal{B} = \mathrm{ind}_{\mathbb{Z}}^{\mathbb{R}}(\mathcal{E}, \alpha)$, and the geometrical PV sequence, and supposing that $\epsilon : \mathbb{R}^{d-1} \to \mathrm{Aut}(\mathcal{E})$ $(d-1 = \dim(\mathcal{E}))$, the T-duals are:

$$\begin{aligned} \widehat{\mathcal{E}} &:= \mathcal{E} \rtimes_{\epsilon} \mathbb{R}^{d-1}, \\ \widehat{\mathcal{B}} &:= \mathcal{B} \rtimes_{\beta} \mathbb{R}^{d} \end{aligned}$$

where $\beta := \tau \times \epsilon : \mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1} \to \mathrm{Aut}(\mathcal{E})$, with τ the translation automorphism $\mathcal{B} = \mathrm{ind}_{\mathbb{Z}}^{\mathbb{R}}(\mathcal{E}, \alpha)$:

$$(\tau(t)f)(x) = f(x-t).$$

Can we reconstruct the physical PV sequence and Toeplitz algebra?

From

$$\begin{aligned}\widehat{\mathcal{E}} &:= \mathcal{E} \rtimes_{\epsilon} \mathbb{R}^{d-1},\\ \widehat{\mathcal{B}} &:= \mathcal{B} \rtimes_{\beta} \mathbb{R}^{d},\end{aligned}$$

we get

$$\begin{aligned}\widehat{\mathcal{B}} &:= \mathcal{B} \rtimes_{\beta} \mathbb{R}^{d}\\ &= \operatorname{ind}_{\mathbb{Z}}^{\mathbb{R}}(\mathcal{E}, \alpha) \rtimes_{\beta} \mathbb{R}^{d}\\ &= \operatorname{ind}_{\mathbb{Z}}^{\mathbb{R}}(\mathcal{E} \rtimes_{\epsilon} \mathbb{R}^{d-1}, \alpha) \rtimes_{\alpha} \mathbb{R}\\ &= \operatorname{ind}_{\mathbb{Z}}^{\mathbb{R}}(\widehat{\mathcal{E}}, \alpha) \rtimes_{\alpha} \mathbb{R}.\end{aligned}$$

Green's Theorem:

$$\mathrm{ind}_{\mathbb{Z}}^{\mathbb{R}}(\mathcal{A},\alpha) \rtimes_{\tau} \mathbb{R} \cong (\mathcal{A} \rtimes_{\alpha} \mathbb{Z}) \otimes \mathcal{K}(L^2(\mathbb{R}/\mathbb{Z}))$$

which is Morita equivalent to $\mathcal{A} \rtimes_{\alpha} \mathbb{Z}$,

so $\widehat{\mathcal{B}} \cong \widehat{\mathcal{E}} \rtimes_{\alpha} \mathbb{Z}$, as asserted.

Connes' Thom Isomorphism Theorem: There is a natural transformation of functors giving

$$K_j(\mathcal{A} \rtimes_\alpha \mathbb{R}^D) \cong K_{j+D}(\mathcal{A}).$$

$$K_j(\widehat{\mathcal{E}}) = K_j(\mathcal{E} \rtimes_\epsilon \mathbb{R}^{d-1}) \cong K_{j+d-1}(\mathcal{E}),$$
$$K_j(\widehat{\mathcal{B}}) = K_j(\mathcal{B} \rtimes_\epsilon \mathbb{R}^{d}) \cong K_{j+d}(\mathcal{B}).$$

Paschke's gloss on the Connes' Thom theorem showed how to reconstruct the Toeplitz algebra.

Using

$$K_j(\widehat{\mathcal{E}}) \cong K_{j+d-1}(\mathcal{E}), \qquad K_j(\widehat{\mathcal{B}}) \cong K_{j+d}(\mathcal{B}),$$

and the geometric PV hexagon

$$\begin{array}{ccccc} K_1(\mathcal{E}) & \longrightarrow & K_1(\mathcal{E}) & \longrightarrow & K_0(\mathcal{B}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{B}) & \longleftarrow & K_0(\mathcal{E}) & \longleftarrow & K_0(\mathcal{E}). \end{array}$$

we obtain (with Bott periodicity):

$$\begin{array}{ccccc} K_d(\widehat{\mathcal{E}}) & \longrightarrow & K_d(\widehat{\mathcal{E}}) & \longrightarrow & K_d(\widehat{\mathcal{B}}) \\ \uparrow & & & & \downarrow \\ K_{d+1}(\widehat{\mathcal{B}}) & \longleftarrow & K_{d+1}(\widehat{\mathcal{E}}) & \longleftarrow & K_{d+1}(\widehat{\mathcal{E}}), \end{array}$$

the physical PV sequence (since $K_j(\mathcal{T}(\widehat{\mathcal{E}})) \cong K_j(\widehat{\mathcal{E}})$).

What about H-flux mathematically?

Dixmier–Douady Theorem (1963). For every locally compact space P and $\delta \in H^3(P, \mathbb{Z})$ there is a C*-algebra $\mathcal{A} = CT(P, \delta)$ (a continuous trace algebra) with spectrum P and Dixmier–Douady obstruction δ, and it is unique up to Morita equivalence, ie all such algebras have the same representation theory.

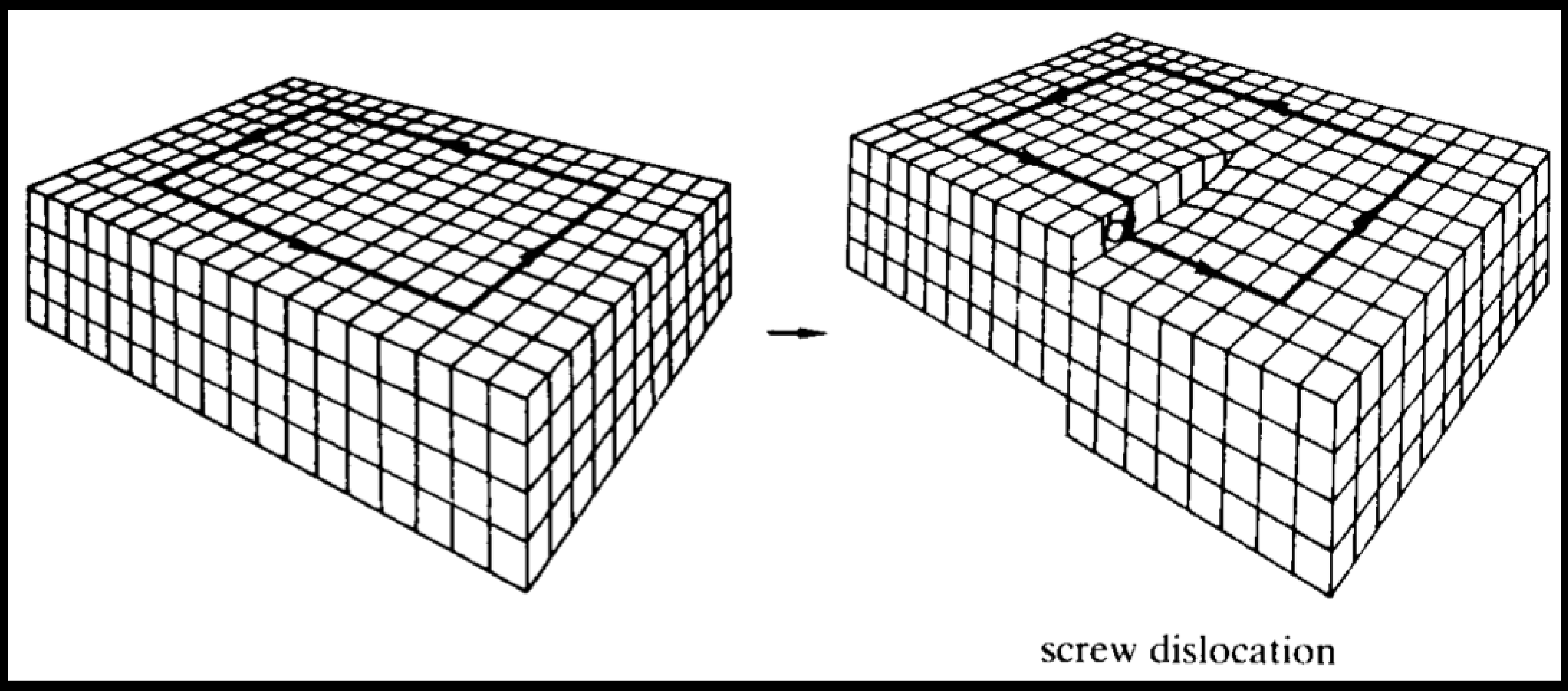
A continuous trace algebra CT(P, δ) may be thought of as an algebra of sections of a compact operator bundle over P.

The Dixmier–Douady class may be thought of as the H-flux through P.

What about H-flux physically?

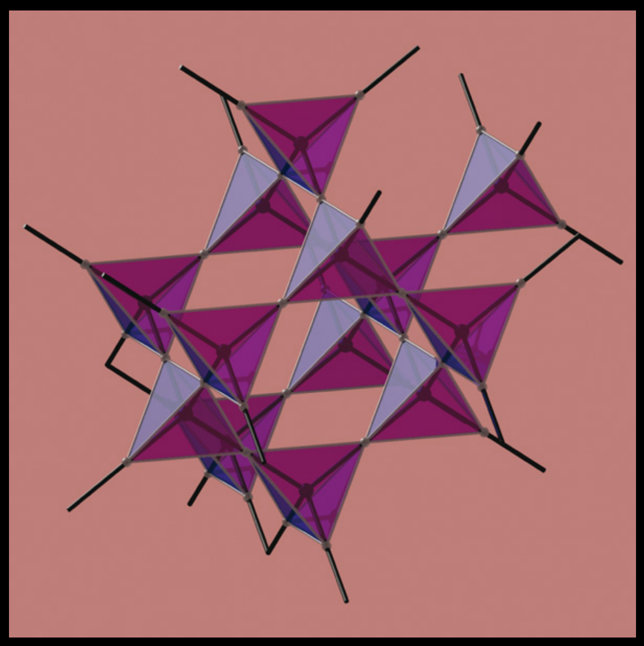
One feature in T-duality is that one usually has an H-flux $H \in H^3(X, \mathbb{Z})$, which is not prominent in condensed matter problems.

Screw dislocations in crystals:

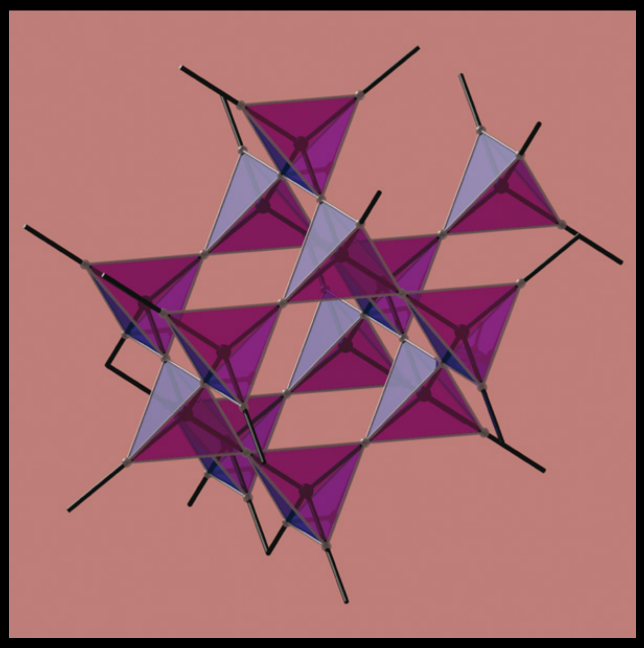


Wild speculation

Could it feature in the spin ice analogues of magnetic monopoles found recently (Castelnovo et al. *Nature* **49** 2008)?



Dipoles at the tetrahedron vertices point either in or out, and normally there are two of each, but one can create anomalous regions with an imbalance of inward and outward pointing dipoles in pyrochlore lattices such as $Dy_2Ti_2O_7$.



The End