Spin^c, K-homology and Proper Actions

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Setting

G an **almost-connected** Lie group.

X a smooth manifold on which G acts **properly** and **cocompactly**.

"Properly" means

$$G \times X \to X \times X$$

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"Cocompactly" means X/G is compact. There are three main results with two applications. We will focus on one: G-equivariant Poincaré duality.

G-equivariant Poincaré duality

Example: (Poincaré duality in the non-equivariant setting) Let M be a compact smooth manifold. Then

$$H_k(M) \cong H^{n-k}(M), \quad n = dim(M).$$

We prove an analogous result in the setting of K-theory. To introduce this, recall the construction of K-homology.

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Example: Let E be a complex vector bundle over a Spin manifold X. Take $H = L^2(S \otimes E)$, ϕ multiplication by functions in $C_0(X)$, F a normalised Spin-Dirac operator $D(1 + D^2)^{-1/2}$.

If K is a compact Lie group acting on a compact manifold Y, it is known from [Kasparov, 1988] that

$$K_\bullet^K(C_\tau(Y)) \cong K_\bullet^K(Y),$$

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We generalise this to:

Theorem (HG, Mathai, Wang '16)

Let G be an almost-connected Lie group acting properly and cocompactly on a smooth manifold X. Then

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3. Decompose $TX \cong G \times_K (TY \times \mathfrak{p})$, where $\mathfrak{p} \oplus \mathfrak{k} = \mathfrak{g}$.
4. A result of [Kasparov, 2015] tells us that the C^*-algebras $C_\tau(X)$ and $C_0(TX)$ have the same K-theory.

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Using equivariant Poincaré duality in the compact group case, get

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Finally, we showed that there is an **induction map** on equivariant analytic K-homology that is an isomorphism (defined with the help of KK theory):

$$\mathrm{Ind}_K^G : K_\bullet^K(Y) \xrightarrow{\simeq} K_{\bullet+d}^G(X).$$

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It follows that

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Compare also with the Connes-Kasparov isomorphism ("Dirac Induction")

$$R(K) \to K_0(C_r^* G).$$

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The top and bottom arrows are (equivariant versions of) certain natural maps defined by taking a manifold M and a vector bundle E over M to a Dirac operator twisted by E.

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Using this, and by showing that Ind'^G_K and Ind^G_K are isomorphisms, we now know that that the top arrow is an isomorphism.

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