Classification of topological phases via coarse topology

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Topological phase of periodic gapped systems

- $\mathcal{H}$: Hilbert space,
- $U : G = \mathbb{Z}^d \curvearrowright \mathcal{H}$: unitary representation

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- $\mathcal{H} := L^2(\mathbb{Z}^d, \mathbb{C}^N)$: Hilbert space,
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Assumption

The Hamiltonian H has a spectral gap at $\mu \in \mathbb{R}$.

We say that H_1 and H_2 are in the same *topological phase* if $E_{\leq\mu}(H_1) \cong E_{\leq\mu}(H_2)$ as vector bundles.

The K^0-group

$\mathrm{K}^0(X) := \mathcal{G}(\mathrm{Vect}_{\mathbb{C}}(X))$ (the group completion). Therefore,

$$f : (\text{topological phases}) \cong \mathrm{Vect}_{\mathbb{C}}(\mathbb{T}^d) \to \mathbb{R}$$

which is additive ($f(H_1 \oplus H_2) = f(H_1) + f(H_2)$),

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Example: The first Chern number for $d = 2$;

$$c_1(E_{\leq \mu}(H)) := \frac{-1}{2\pi i} \int_{\mathbb{T}^2} \mathrm{tr}(p_x[\nabla_1, p_x][\nabla_2, p_x])dx$$

(p_x: orthogonal projection onto $E_{\leq \mu}(H)_x$).

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Rem. In 2d IQHE, it is related to the Hall conductance by the TKNN formula.

Main result

Two Questions

1. What is the relevant observable algebra for non-periodic systems?
2. How to deal with symmetry of quantum mechanics?

"Theorem" (K.'16)

- The **twisted equivariant** K_0**-group** of the **uniform Roe algebra** classifies topological phases controlled over X.
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(1)Algebra of observables

In the study of periodic systems, the algebra of observables is $C(\mathbb{T}^d, \mathbb{M}_N) \cong C_r^*(\mathbb{Z}^d) \otimes \mathbb{M}_N$ (whose K_0-group is $K^0(\mathbb{T}^d)$).

For the classification of all topological phases, we need the C^*-algebra A containing all possible observables. In particular, it should contain

$$H + V$$

for all $V \in c_b(\mathbb{Z}^d, \mathbb{M}_N)$.

The smallest C^*-algebra containing $C_r^*(\mathbb{Z}^d) \otimes \mathbb{M}_N$ and all potential functions is the 'crossed product' $\mathbb{Z}^d \ltimes c_b(\mathbb{Z}^d, \mathbb{M}_N)$.

Although it is too big (not even separable) to apply some functional analysis, we can study its topology from the viewpoint of metric space geometry.

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Coarse C^*-algebras

Let X be a discrete metric space and $\mathcal{H} := \ell^2(X)$.

Definition

We say that

- $T \in \mathbb{B}(\mathcal{H})$ is *controlled* if $\exists R > 0$ s.t. $T_{xy} = 0$ for $d(x, y) > R$,
- $T \in \mathbb{B}(\mathcal{H}^\infty)$ is *locally compact* if $T\delta_x, \delta_x T \in \mathbb{K}$,

$$C_u^*(X) := \overline{\{T \in \mathbb{B}(\mathcal{H}) \mid \text{controlled }\}}$$
$$C^*(X) := \overline{\{T \in \mathbb{B}(\mathcal{H}^\infty) \mid \text{controlled, locally compact}\}}$$

Then,

$$C_u^*(|\mathbb{Z}^d|) \cong \mathbb{Z}^d \ltimes c_b(\mathbb{Z}^d), C^*(|\mathbb{Z}^d|) \cong \mathbb{Z}^d \ltimes c_b(\mathbb{Z}^d, \mathbb{K})$$

Rem. $c_b(\mathbb{Z}^d) \otimes \mathbb{K} \neq c_b(\mathbb{Z}^d, \mathbb{K})$.

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Rem. $c_b(\mathbb{Z}^d) \otimes \mathbb{K} \neq c_b(\mathbb{Z}^d, \mathbb{K})$.

Coarse Mayer-Vietoris sequence

Let $X := \mathbb{Z}^n$, $Y_\pm := \mathbb{Z}^{n-1} \times \mathbb{Z}_\pm$ and $Z := Y_+ \cap Y_- = \mathbb{Z}^{n-1}$. Then, we get the following Mayer-Vietoris type exact sequence

$$\cdots \to \mathrm{K}_*(C^*_u(Z)) \to \mathrm{K}_*(C^*_u(Y_+)) \oplus \mathrm{K}_*(C^*_u(Y_-)) \to \mathrm{K}_*(C^*_u(X)) \xrightarrow{\partial_{\mathrm{MV}}} \mathrm{K}_{*-1}(C^*_u(Z)) \to \mathrm{K}_{*-1}(C^*_u(Y_+)) \oplus \mathrm{K}_{*-1}(C^*_u(Y_-)) \to \cdots$$

The boundary map $\partial_{\mathrm{MV}} \colon \mathrm{K}_0(C^*_u(X)) \to \mathrm{K}_{-1}(C^*_u(Z))$ is given by

$$[p] \mapsto \partial[\pi(P_+ p P_+)]$$

where P_+ is the projection onto $\ell^2(Y_+)$ and ∂ is the boundary map associated with

$$0 \to C^*_u(Z \subset Y_+) \to C^*_u(Y_+) \to C^*_u(Y_+)/C^*_u(Z \subset Y_+) \to 0$$

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Quasi-crystals

$X \subset \mathbb{R}^d$ is a *Delone set* if it is

- uniformly discrete i.e. $\exists r > 0$ s.t. $B(r, x) \cap X \leq 1$ for $\forall x \in \mathbb{R}^d$ and
- relatively dense i.e. $\exists R > 0$ s.t. $B(R, x) \cap X \geq 1$ for $\forall x \in \mathbb{R}^d$.

Lemma

The above X is a proper metric space with bounded geometry and $C^*_u X \otimes \mathbb{K} \cong C^*_u |\mathbb{Z}^d| \otimes \mathbb{K}$.

(2)Symmetry of quantum mechanics

Let $\mathcal{H}$, U, H be as in IQHE.
In the case of type AII topological insulators, we also assume that $\exists T : \mathcal{H} \to \mathcal{H}$ s.t.

- T is antilinear,
- $TU_g = U_g T$ and $T = (T_k \colon \mathcal{H}_k \to \mathcal{H}_{-k})_{k \in \mathbb{T}^2}$ is continuous ,
- $TH = HT$, $T^2 = -1$,

Then, the projection $E_\mu(H)$ is a "quartanionic vector bundle" on $\mathbb{T}^2$ (with the real structure $\tau \colon k \mapsto -k$) and hence an element in $\mathrm{KQ}^0(\mathbb{T}^2, \tau)$.

(2)Symmetry of quantum mechanics

Let $\mathcal{H}$, U, H be as in IQHE.
In the case of type AIII topological insulators, we also assume that $\exists P : \mathcal{H} \to \mathcal{H}$ s.t.

- P is linear,
- $PU_g = U_g P$ and $P = (P_k : \mathcal{H}_k \to \mathcal{H}_k)_{k \in \mathbb{T}^2}$ is continuous,
- $PH = -HP$, $P^2 = 1$,

Then, the pair $(H|H|^{-1}, P)$ $(+\alpha)$ determines a "chiral vector bundle" (Nittis-Gomi'15) on $\mathbb{T}^2$ and hence an element in $\mathrm{K}^1(\mathbb{T}^2)$.

Wigner's theorem

$\mathcal{H}$: $\mathbb{Z}_2$-graded separable Hilbert space.

$\to \mathbb{P}\mathcal{H} := (\mathcal{H} \setminus \{0\})/\mathbb{T}$: the space of states.

It is equipped with the function

$$\Phi(\_,\_) : \mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H} \to \mathbb{R}_{>0}, \Phi([\xi],[\eta]) = \frac{|\langle \xi, \eta \rangle|}{\|\xi\| \, \|\eta\|}.$$

The group of symmetries in quantum mechanics:

$$\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H}) := \{ f : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H} \mid f^*\Phi = \Phi, f\gamma = \gamma f \}$$

Theorem (Wigner's theorem)

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Twists

Definition

A symmetry of quantum mechanics is a group homomorphism $G \to \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$.

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Theorem (Freed-Moore'13, K.'16)

The data (ϕ, c, τ) is classified by the set

$$\coprod_{\phi \in \check{H}^1(G;\mathbb{Z}_2)} \check{H}^1(G;\mathbb{Z}_2) \ltimes_{\epsilon} \check{H}^2(G;\mathbb{T}).$$

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The twisted equivariant K-group

- G: finite group,
- (ϕ, c, τ): twist on G,
- A: ϕ-twisted ($\mathbb{Z}_2$-graded) G-C^*-algebra i.e. $G \curvearrowright A$ s.t. α_g is linear/antilinear if $\phi(g) = 0/1$.

We define the twisted equivariant K-group ${}^\phi\mathrm{K}^G_{*,c,\tau}(A)$ for these data. It gives a functor

$$ {}^\phi\mathrm{K}^G_{*,c,\tau} \colon {}^\phi\mathfrak{Calg}^G_{\mathbb{Z}_2} \to \mathfrak{Ab}, $$

which is a canonical generalization of K^G_* and KR^G_*.

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The twisted equivariant K_0-group

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Assume the $\mathbb{Z}_2$-grading of A is trivial.

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Set

$$\mathcal{F}^{G}_{c,\mathcal{V}}(A) := \{ s \in A \hat{\otimes} \mathbb{K}(\mathcal{V})_{\mathrm{sa}} \mid s^2 = 1, \alpha_g(s) = (-1)^{c(g)} s \}$$

Theorem

$${}^{\phi}\mathrm{K}^{G}_{0,c,\tau}(A) = \bigcup_{\mathcal{V}} \mathcal{F}^{G}_{c,\mathcal{V}}(A) / \sim_{\mathrm{homotopy}}$$

H with $HU_g = (-1)^{c(g)} U_g H \Rightarrow [H|H|^{-1}] \in {}^{\phi}\mathrm{K}^{G}_{0,c,\tau}(A)$.

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We say that $\mathcal{V}$ is a (ϕ, c, τ)-twisted representation of G if $\mathcal{V}$ is a $\mathbb{Z}_2$-graded vector space with ϕ-linear, c-graded and τ-projective representation of G.

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The twisted equivariant K_0-group

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Main result

Let

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Definition-Theorem

- We say that $\mathcal{TP}_b(X; G, \phi, c, \tau) := {}^{\phi}\mathrm{K}^G_{0,c,\tau}(C^*_u(X))$ is the set of bulk topological phases.
- We say that $\mathcal{TP}_e(Z \subset Y_+; G, \phi, c, \tau) := \mathrm{Im}\,\partial \subset {}^{\phi}\mathrm{K}^G_{-1,c,\tau}(C^*_u(Z))$ is the set of edge topological phases.
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$$
\begin{array}{ccccc}
{}^{\phi}\mathrm{K}^{G}_{0,c,\tau}(C^*_u(X)) & \longrightarrow & {}^{\phi}\mathrm{K}^{G}_{0,c,\tau}(C^*(X)) & & \\
\Big\downarrow \partial_{\mathrm{MV}} & & \Big\downarrow \partial_{\mathrm{MV}} & \searrow^{\mathrm{cBC}} & \\
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Example: CT-symmetries

We consider the case that $(\phi, c) : \mathcal{A} \to \mathbb{Z}_2 \times \mathbb{Z}_2$ is injective. Choices of $(\mathcal{A}, \tau)$ are classified by

$$C^1 = \pm 1 \text{ and } T^2 = \pm 1$$

$(C, T \in \mathcal{A}^\tau$ are lifts of $(1,1), (1,0) \in \mathcal{A}$ s.t. $(CT)^2 = 1)$.
There are 10 choices of such $(\mathcal{A}, \tau)$. For each of them, ${}^\phi\mathrm{K}^{\mathcal{A}}_{0,c,\tau}(\_)$ coincides with K_* or KR_* as following.

$\mathcal{A}$	1	$\mathcal{P}$	$\mathcal{T}$		$\mathcal{C}$		$\mathcal{G}$			
C^2					1	-1	1	1	-1	-1
T^2			1	-1			1	-1	1	-1
${}^\phi C^*_{c,\tau}\mathcal{A}$	$\mathbb{C}$	$\mathbb{C}\ell_1$	$\mathbb{M}_2(\mathbb{R})$	$\mathbb{H}$	$\mathbb{C}\ell_{0,2}$	$\mathbb{C}\ell_{2,0}$	$\mathbb{C}\ell_{1,2}$	$\mathbb{C}\ell_{0,3}$	$\mathbb{C}\ell_{2,1}$	$\mathbb{C}\ell_{3,0}$
${}^\phi\mathrm{K}^{\mathcal{A}}_{0,c,\tau}$	K_0	K_1	KR_0	KR_4	KR_2	KR_6	KR_1	KR_3	KR_7	KR_5
Cartan	A	AIII	AI	AII	D	C	BDI	DIII	CI	CII

Table: The 10-fold way and Clifford algebras

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Table: The 10-fold way and Clifford algebras

dim	A	AIII	AI	BDI	D	DIII	AII	CII	C	CI
0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0
2	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
3	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$

Table: Kitaev's periodic table

cf. Bott periodicity

$$\pi_i(U) \cong \begin{cases} \mathbb{Z} & i = 2n+1 \\ 0 & i = 2n \end{cases}, \pi_i(O) \cong \begin{cases} \mathbb{Z} & i = 8n-1, 8n+3 \\ \mathbb{Z}_2 & i = 8n, 8n+1 \\ 0 & \text{otherwise} \end{cases}$$

Example: reflection-invariant systems

$G = \mathcal{A} \times \mathcal{R}$, where $\mathcal{R} \cong \mathbb{Z}_2$ acting on the material as a reflection. Choices of (G, τ) is classified by

$$C^2 = \pm 1,\ T^2 = \pm 1,\ TR = \pm RT,\ PR = \pm RP$$

($P := CT$, R is the lift of the generator of $\mathcal{R}$ s.t. $R^2 = 1$).
It is not difficult to determine the finite-dimensional algebras $G \ltimes^{\phi}_{c,\tau} \mathrm{C}\ell_{0,d}$ and we get

$${}^{\phi}\mathrm{K}^{G}_{0,c,\tau}(\mathrm{C}\ell_{0,d}) \cong \begin{cases} {}^{\phi}\mathrm{K}^{\mathcal{A}}_{d-1,c,\tau}(\mathbb{R}) & \text{if } (\epsilon,\nu) = (+,+), \\ {}^{\phi}\mathrm{K}^{\mathcal{A}}_{d+1,c,\tau}(\mathbb{R}) & \text{if } (\epsilon,\nu) = (+,-), \\ {}^{\phi}\mathrm{K}^{\mathcal{A}}_{d,c,\tau}(\mathbb{R})^2 & \text{if } (\epsilon,\nu) = (-,+), \\ \mathrm{K}_{d,c,\tau}(\mathbb{R}) & \text{if } (\epsilon,\nu) = (-,-). \end{cases}$$

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Reflection	Class	C_q or R_q	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
R	A	C_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
R^+	AIII	C_0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
R^-	AIII	C_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	AI	R_1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
	BDI	R_2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
	D	R_3	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
R^+, R^{++}	DIII	R_4	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	AII	R_5	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	CII	R_6	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	C	R_7	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	CI	R_0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	AI	R_7	0	0	0	$\mathbb{Z}$	0	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$	$\mathbb{Z}$
	BDI	R_0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$
	D	R_1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	"$\mathbb{Z}_2$"
R^-, R^{--}	DIII	R_2	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
	AII	R_3	0	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
	CII	R_4	$\mathbb{Z}$	0	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	C	R_5	0	$\mathbb{Z}$	0	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	CI	R_6	0	0	$\mathbb{Z}$	0	"$\mathbb{Z}_2$"	$\mathbb{Z}_2$	$\mathbb{Z}$	0
R^{+-}	BDI	R_1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
R^{-+}	DIII	R_3	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
R^{+-}	CII	R_5	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
R^{-+}	CI	R_7	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
R^{-+}	BDI, CII	C_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
R^{+-}	DIII, CI	C_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

Classification of reflection invariant topological phases
Takahiro Morimoto and Akira Furusaki, Topological classification with additional symmetries from Clifford algebras, Phys. Rev. B 88, 125129.

Example: $1D$ type A reflection invariant systems

$$\mathrm{ind}\colon \mathrm{K}^0_{\mathcal{R}}(\mathbb{T}^1) \to \mathrm{K}_0^{\mathcal{R}}(\mathrm{C}\ell_{0,1}) \cong \mathbb{Z}.$$

The simplest vector bundle with nontrivial index is $E \to \mathbb{T}^1$ s.t. $E|_0 \cong V_+$ and $E|_\pi \cong V_-$ ($V_\pm \cong \mathbb{C}$ with the $\mathbb{Z}_2$-action given by ± 1).
The corresponding Hamiltonian is

$$H := \frac{1}{2}\begin{pmatrix} s+s^* & i(s-s^*) \\ i(s-s^*) & -(s+s^*) \end{pmatrix} \in \mathbb{B}(\ell^2(\mathbb{Z}; V_+ \oplus V_-)),$$

where s is the shift operator.

cf.) the clean Kitaev chain (a $1D$ type BDI systems):

$$H = \frac{1}{2}\begin{pmatrix} s+s^*+2\mu & -i(s-s^*) \\ -i(s-s^*) & -(s+s^*+2\mu) \end{pmatrix},$$

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