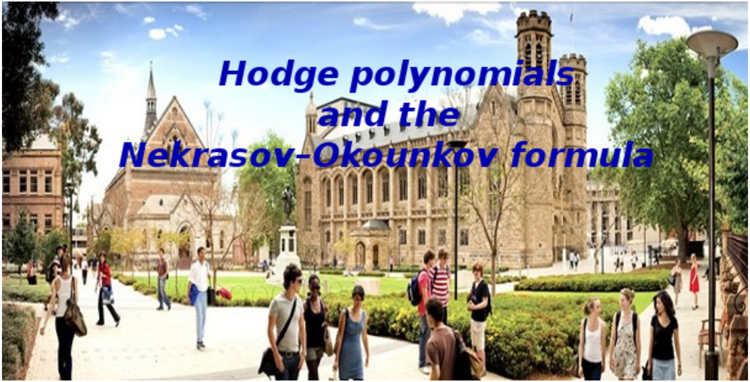


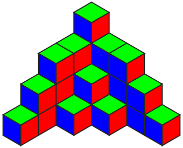
Topological Matter, Strings, K-theory and related areas

26–30 September 2016

This talk is based on joint work with  from Caltech.

Outline

1. A string theorist's view of



2. Mixed Hodge polynomials associated with

The infinite wedge space

Let V be a linear space with basis $\{\underline{k}\}_{k\in\mathbb{Z}+\frac{1}{2}}$

A semi-infinite monomial v_S is an expression of the form

$$v_S = \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \cdots$$

where the $s_i \in \mathbb{Z} + \frac{1}{2}$, $s_i > s_{i+1}$ and $s_i - s_{i+1} = 1$ for $i \gg 1$.

We say that v_S has charge c if

$$s_i = c - i + \frac{1}{2} \quad \text{for } i \gg 1$$

Examples:

$$\underline{-\tfrac{1}{2}} \wedge \underline{-\tfrac{3}{2}} \wedge \underline{-\tfrac{5}{2}} \wedge \underline{-\tfrac{7}{2}} \wedge \underline{-\tfrac{9}{2}} \wedge \cdots \qquad c = 0$$

$$\underline{\tfrac{7}{2}} \wedge \underline{\tfrac{3}{2}} \wedge \underline{-\tfrac{1}{2}} \wedge \underline{-\tfrac{3}{2}} \wedge \underline{-\tfrac{5}{2}} \wedge \cdots \qquad c = 2$$

and

$$\underline{\tfrac{5}{2}} \wedge \underline{-\tfrac{1}{2}} \wedge \underline{-\tfrac{7}{2}} \wedge \underline{-\tfrac{9}{2}} \wedge \underline{-\tfrac{11}{2}} \wedge \cdots \qquad c = -1$$

The infinite wedge space (or fermionic Fock space) $\Lambda^{\frac{\infty}{2}} V$ is the linear space with basis $\{v_S\}$, equipped with an inner product for which this basis is orthonormal.

The wedging operator ψ_k, $k \in \mathbb{Z} + \frac{1}{2}$ is defined by

$$\psi_k : \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V, \qquad f \mapsto \underline{k} \wedge f$$

Together with its adjoint, the contraction operator ψ_k^* (which "sign-removes $\underline{k}$"), this yields the anti-commutation relations of the infinite Clifford algebra:

$$\{\psi_k, \psi_l^*\} = \delta_{kl}, \qquad \{\psi_k, \psi_l\} = \{\psi_k^*, \psi_l^*\} = 0$$

Obviously,

$$\psi_k \psi_k^*(v_S) = \begin{cases} v_S & \text{if } k \in S \\ 0 & \text{otherwise} \end{cases} \qquad \psi_k^* \psi_k(v_S) = \begin{cases} v_S & \text{if } k \notin S \\ 0 & \text{otherwise} \end{cases}$$

Using the free fermions ψ_k and ψ_k^* one can further define the free bosons

$$\alpha_n = \sum_{k \in \mathbb{Z}+\frac{1}{2}} \psi_{k-n}\psi_k^* \qquad n \in \mathbb{Z} \setminus \{0\}$$

with Heisenberg commutation relations

$$[\alpha_n, \alpha_m] = n\delta_{n,-m}$$

and adjoint $a_n^* = a_{-n}$.

Finally we use these to define the vertex operators

$$\Gamma_\pm(z) = \exp\left(\sum_{n \geqslant 1} \frac{z^n}{n}\,\alpha_{\pm n}\right)$$

Partitions

The infinite wedge space $\Lambda^{\frac{\infty}{2}} V$ is a direct sum of charge-c subspaces

$$\Lambda^{\frac{\infty}{2}} V = \bigoplus_{c \in \mathbb{Z}} \left(\Lambda^{\frac{\infty}{2}} V\right)_c$$

The semi-infinite monomials spanning each subspace are in one-to-one correspondence with integer partitions: If

$$v_S = \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \cdots$$

has charge c then the partition corresponding to v_S is

$$\lambda = (\lambda_1, \lambda_2, \dots) \quad \text{where} \quad \lambda_i = s_i + i - c - \tfrac{1}{2}$$

<u>Examples</u>:

$$c = 0 \qquad \underline{-\tfrac{1}{2}} \wedge \underline{-\tfrac{3}{2}} \wedge \underline{-\tfrac{5}{2}} \wedge \underline{-\tfrac{7}{2}} \wedge \underline{-\tfrac{9}{2}} \wedge \cdots \qquad \lambda = 0$$

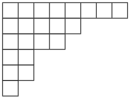
$$c = 2 \qquad \underline{\tfrac{7}{2}} \wedge \underline{\tfrac{3}{2}} \wedge \underline{-\tfrac{1}{2}} \wedge \underline{-\tfrac{3}{2}} \wedge \underline{-\tfrac{5}{2}} \wedge \cdots \qquad \lambda = (2, 1)$$

and

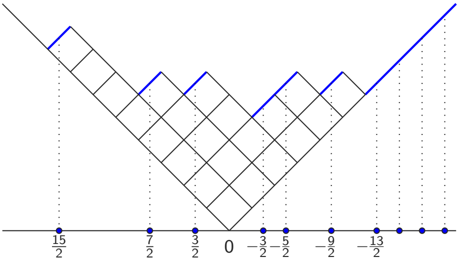
$$c = -1 \qquad \underline{\tfrac{5}{2}} \wedge \underline{-\tfrac{1}{2}} \wedge \underline{-\tfrac{7}{2}} \wedge \underline{-\tfrac{9}{2}} \wedge \underline{-\tfrac{11}{2}} \wedge \cdots \qquad \lambda = (3, 2)$$

For $c = 0$ Okounkov introduced the following graphical description obtained by rotating a partition, such as

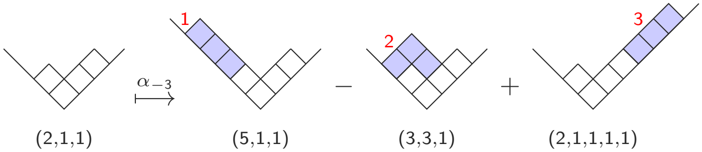
$$(8, 5, 4, 2, 2, 1) \; =$$



by 135° to get



Using partitions to represent the charge-0 semi-infinite monomials and adopting "bra" and "ket" notation, for $n \geqslant 1$ the operators α_{-n} and α_n act on $|\lambda\rangle$ by adding/deleting border strips of length n, weighted by the factor $(-1)^{h+1}$ where h is the height of the border strip.



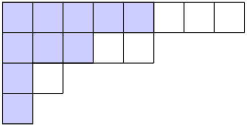
$$\alpha_{-3}\big|(2,1,1)\big\rangle = \big|(5,1,1)\big\rangle - \big|(3,3,1)\big\rangle + \big|(2,1,1,1,1)\big\rangle$$

Accordingly, the vertex operators $\Gamma_\pm(z)$ satisfy

$$\Gamma_-(z)|\mu\rangle = \sum_{\lambda \succ \mu} z^{|\lambda-\mu|}|\lambda\rangle \quad \text{and} \quad \Gamma_+(z)|\lambda\rangle = \sum_{\mu \prec \lambda} z^{|\lambda-\mu|}|\mu\rangle$$

where a pair of partitions λ, μ is interlacing, denoted as $\lambda \succ \mu$, if

$$\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \cdots$$



$$(8,5,2,1) \succ (5,3,1,1)$$

Schur functions

The Schur functions $s_\lambda(x_1, x_2, \ldots, x_n)$ are the characters of the irreducible polynomial representations of $\mathrm{GL}(n, \mathbb{C})$ of highest weight λ.

For λ, μ partitions such that $\mu \subseteq \lambda$, the skew Schur function $s_{\lambda/\mu}(x)$ may be computed combinatorially as

$$s_{\lambda/\mu}(x) = \sum_T x^T$$

where the sum is over semi-standard Young tableaux on $\{1, 2, 3, \ldots, n\}$ of skew shape λ/μ.

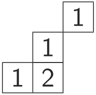
For example

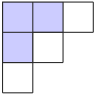
$$s_{(3,2,2)/(2,1)}(x_1, x_2) \text{ “}=\text{” } \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & \\ \hline 1 & 2 & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & 2 \\ \hline & 1 & \\ \hline 2 & 2 & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & 2 \\ \hline & 1 & \\ \hline 1 & 2 & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & \\ \hline 2 & 2 & \\ \hline \end{array}$$

$$= x_1 x_2 (x_1^2 + x_2^2) + 2 x_1^2 x_2^2$$

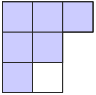
Since a semi-standard Young tableaux of shape λ/μ on $\{1, 2, \dots, n\}$ is in one-to-one correspondence with sequences of interlacing partitions

$$\mu = \lambda^{(0)} \prec \lambda^{(1)} \prec \lambda^{(2)} \prec \cdots \prec \lambda^{(n)} = \lambda$$





(2,1)≺(3,2,1)



(3,2,1)≺(3,2,2)

$$\lambda^{(0)} = (2, 1) = \mu$$
$$\lambda^{(1)} = (3, 2, 1)$$
$$\lambda^{(2)} = (3, 2, 2) = \lambda$$

we have

$$s_{\lambda/\mu}(x) = \sum_{\substack{\lambda^{(0)} \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(n)} \\ \lambda^{(0)} = \mu \\ \lambda^{(n)} = \lambda}} x_1^{|\lambda^{(1)} - \lambda^{(0)}|} x_2^{|\lambda^{(2)} - \lambda^{(1)}|} \cdots x_n^{|\lambda^{(n)} - \lambda^{(n-1)}|}$$

Hence

$$\left\langle \lambda \middle| \prod_{i \geqslant 1} \Gamma_-(x_i) \middle| \mu \right\rangle = \left\langle \mu \middle| \prod_{i \geqslant 1} \Gamma_+(x_i) \middle| \lambda \right\rangle = s_{\lambda/\mu}(x_1, x_2, \dots)$$

3-dimensional partitions

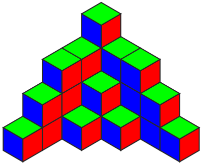
A plane partition or 3-dimensional partition is a two-dimensional array of nonnegative integers such that the numbers are weakly decreasing from left to right and from top to bottom, and such that finitely many numbers are positive.

Geometrically, a plane partition may also be thought of as a configuration of stacked unit cubes, such that . . .

For example,

4	3	3	2	1
3	2	1		
3	1			
2				
1				

and



represent the same plane partition of 26.

A famous result of MacMahon is the following closed-form formula for the generating function of plane partitions

$$\sum_{\pi} q^{|\pi|} = \prod_{n \geqslant 1} \frac{1}{(1-q^n)^n}$$

where $|\pi|$ is the number of unit cubes in π.

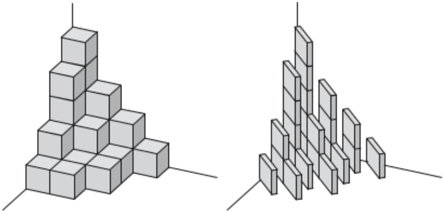
Okounkov and Reshetikhin showed that the above formula follows as a straightforward application of the vertex operators $\Gamma_{\pm}(z)$.

Given a plane partition

5	3	2	1
4	2	1	
2	1	1	
1	1		

we can read off its sequence of diagonal slices to obtain a sequence of interlacing partitions

$$0 \prec (1) \prec (2,1) \prec (4,1) \prec (5,2,1) \succ (3,1) \succ (2) \succ (1) \succ 0$$



Each partition λ in the sequence of diagonal slices contributes $q^{|\lambda|}$ to the weight q^π of π. For this we need the operator

$$Q|\lambda\rangle = q^{|\lambda|}|\lambda\rangle$$

which q-commutes with the vertex operators $\Gamma_\pm(z)$:

$$\Gamma_\pm(z)\, Q = Q\, \Gamma_\pm(zq^{\pm 1})$$

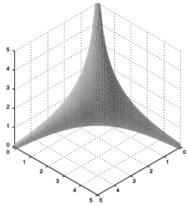
Putting this all together yields

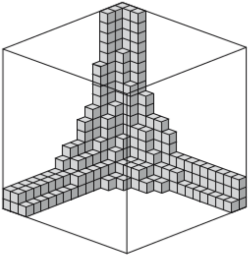
$$\sum_\pi q^\pi = \left\langle 0 \middle| \prod_{i\geqslant 1} \left(\Gamma_+(1) Q\right) \prod_{i\geqslant 1} \left(\Gamma_-(1) Q\right) \middle| 0 \right\rangle$$

$$= \cdots \qquad \Gamma_+(z)\Gamma_-(1/w) = \frac{1}{1-z/w}\,\Gamma_-(z)\Gamma_+(1/w)$$

$$= \prod_{n\geqslant 1} \frac{1}{(1-q^n)^n}$$

In their work on the limit shape of 3-d partitions, Okounkov, Reshetikhin and Vafa introduced the following model for 3-d partitions:





$$\lambda = (3,2) \qquad N_1 = 16$$
$$\mu = (3,1) \qquad N_2 = 16$$
$$\nu = (3,1,1) \qquad N_3 = 16$$

$$P(\lambda,\mu,\nu) := \lim_{N_1,N_2,N_3\to\infty} q^{-N_1|\lambda|-N_2|\mu|-N_3|\nu|} P_{N_1,N_2,N_3}(\lambda,\mu,\nu)$$

Okounkov, Reshetikhin and Vafa first let $N_3 \to \infty$ and then again read off the sequence of diagonal slices, now of the form

$$\lambda' \prec \cdots \succ \mu$$

with possible shapes of the slices determined by the choice of ν. Using the vertex operator formalism, they then show that

$$P(\lambda, \mu, \nu) = \frac{q^{-n(\lambda')-n(\mu)-n(\nu')}}{\prod_{n \geqslant 1}(1-q^n)^n} \times s_{\nu'}(q^\rho) \sum_\eta q^{-|\eta|} s_{\lambda'/\eta}(q^{-\nu+\rho}) s_{\mu/\eta}(q^{-\nu'+\rho})$$

where $n(\lambda) = \sum_{i \geqslant 1}(i-1)\lambda_i$, $\rho = (0, 1, 2, \dots)$ and

$$f(q^{-\lambda+\rho}) = f(q^{-\lambda_1+0}, q^{-\lambda_2+1}, q^{-\lambda_3+2}, \dots)$$

For $\lambda = \mu = \nu = 0$ this simplifies to MacMahon's formula.

The Nekrasov–Okounkov formula

The topological vertex $C_{\lambda\mu\nu}(q)$ was introduced by Aganagic, Klemm, Marino and Vafa to compute Gromov–Witten and Donaldson–Thomas invariants of toric Calabi–Yau threefolds. It may be expressed in terms of skew Schur functions as

$$C_{\lambda\mu\nu}(q) = q^{n(\lambda)-n(\lambda')+n(\nu)-n(\nu')+\frac{1}{2}(|\lambda|+|\mu|+|\nu|)} \times s_{\nu'}(q^{\rho}) \sum_{\eta} q^{-|\eta|} s_{\lambda'/\eta}\big(q^{-\nu+\rho}\big) s_{\mu/\eta}\big(q^{-\nu'+\rho}\big)$$

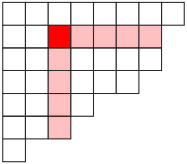
Comparing this with the Okounkov–Reshetikhin–Vafa formula we get

$$C_{\lambda\mu\nu}(q) = q^{n(\lambda)+n(\mu)+n(\nu)+\frac{1}{2}(|\lambda|+|\mu|+|\nu|)} P(\lambda,\mu,\nu) \prod_{n\geqslant 1}(1-q^n)^n$$

Since $P(\lambda,\mu,\nu)$ clearly is cyclically symmetric, we may infer that

$$\boxed{C_{\lambda\mu\nu}(q) = C_{\mu\nu\lambda}(q) = C_{\nu\lambda\mu}(q)}$$

The hook-length $h(s)$ of a square $s \in \lambda$ is the number of squares immediately to the right and below s, including s itself. For example, the square $s = (3, 2) =$ ■ in $(8, 7, 7, 6, 4, 3, 1)$ has hook-length 9.



Using the cyclic symmetry of the topological vertex to compute the sum

$$\sum_{\lambda,\mu} T^{|\lambda|}(-u)^{|\lambda|-|\mu|} C_{0\lambda\mu}(q) C_{0\lambda'\mu'}(q)$$

in two different ways yields

$$\sum_{\lambda} T^{|\lambda|} \prod_{s\in\lambda} \frac{(1-uq^{h(s)})(1-u^{-1}q^{h(s)})}{(1-q^{h(s)})^2} = \prod_{k,r\geqslant 1} \frac{(1-uq^r T^k)^r (1-u^{-1}q^r T^k)^r}{(1-q^{r-1}T^k)^r (1-q^{r+1}T^k)^r}$$

Setting $u = q^z$ and letting q tend to 1 yields the Nekrasov–Okounkov formula for an arbitrary power of the Dedekind η-function

$$\prod_{k\geqslant 1}(1 - T^k)^{z^2-1} = \sum_{\lambda} T^{|\lambda|} \prod_{s\in\lambda}\left(1 - \frac{z^2}{h(s)^2}\right) \qquad z \in \mathbb{C}$$

Mixed Hodge polynomials of character varieties

In the following we are interested in the affine variety

$$\mathcal{M}_n := \{A_1, B_1, \dots, A_g, B_g \in \mathrm{GL}(n, \mathbb{C}) : \\ A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} = \zeta_n I\} // \mathrm{GL}(n, \mathbb{C})$$

where g is a nonnegative integer, ζ_n a primitive nth-root of unity and $//$ a GIT quotient.

$\mathcal{M}_n$ is the twisted character variety of a closed Riemann surface Σ of genus g with points the twisted homomorphisms from $\pi_1(\Sigma)$ to $\mathrm{GL}(n, \mathbb{C})$ modulo conjugation. It is nonsingular of dimension d_n given by

$$d_n = 2n^2(g-1) + 2 \qquad g \geqslant 1$$

Hausel and Rodriguez-Villegas considered the problem of computing the Poincaré polynomials

$$P(\mathcal{M}_n; t) = \sum_i b_i(\mathcal{M}_n) t^i$$

with $b_i(\mathcal{M}_n)$ the Betti numbers of $\mathcal{M}_n$—extending earlier work of Hitchin ($n = 2$) and Gothen ($n = 3$).

$\mathcal{M}_n$ admits a mixed Hodge structure (in the sense of Deligne) on its cohomology which is of "diagonal type". Hence its (mixed) Hodge polynomial, which is a 3-parameter deformation of the Poincaré polynomial, is effectively a 2-variable polynomial, $H(\mathcal{M}_n; q, t)$.

Moreover

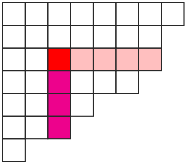
$$P(\mathcal{M}_n; t) = H(\mathcal{M}_n; 1, t)$$
$$E(\mathcal{M}_n; q) = q^{d_n} H(\mathcal{M}_n; 1/q, -1)$$

where $E(\mathcal{M}_n; q)$ is the E-polynomial of $\mathcal{M}_n$, counting the number of points of $\mathcal{M}_n$ when considered over the finite field $\mathbb{F}_q$ instead of $\mathbb{C}$.

More generally, Hausel and Rodriguez-Villegas tried to get a handle on $H(\mathcal{M}_n; q, t)$.

We refine the hook-length of a square $s \in \lambda$ by defining the arm-length $a(s)$ and leg-length $l(s)$ as the number of squares immediately to the right respectively below s, excluding s itself.
Hence $h(s) = a(s) + l(s) + 1$.

For example, the square $s = (3, 3) =$  in $(8, 7, 7, 6, 4, 3, 1)$ has arm-length 4 and leg-length 3.



Defining the function $\overline{H}_n(u,q,t) = \overline{H}_n(u,q,t;g)$ by

$$\sum_{\lambda} T^{|\lambda|} t^{(1-g)(2n(\lambda)+|\lambda|)} \prod_{s\in\lambda} \frac{((1-uq^{a(s)+1}t^{l(s)})(1-u^{-1}q^{a(s)}t^{l(s)+1}))^g}{(1-q^{a(s)+1}t^{l(s)})(1-q^{a(s)}t^{l(s)+1})}$$
$$= \operatorname{Exp}\left(\sum_{n\geqslant 1} \frac{\overline{H}_n(u,q,t)T^n}{(1-q)(t^{-1}-1)}\right)$$

where Exp is a plethystic exponential; if

$$f(u,q,t;T) = \sum_{n\geqslant 1} c_n(u,q,t)T^n$$

then

$$\operatorname{Exp}\left(f(u,q,t;T)\right) = \exp\left(\sum_{n\geqslant 1} \frac{f(u^n,q^n,t^n;T^n)}{n}\right)$$

Example

$$\operatorname{Exp}\left(\frac{T}{1-T}\right) = \prod_{n\geqslant 1} \frac{1}{1-T^n}$$

Conjecture. (Hausel, Rodriguez-Villegas)

The mixed Hodge polynomial of $\mathcal{M}_n$ is given by

$$H(\mathcal{M}_n; q, t) = (q^{1/2} t)^{d_n} \, \overline{H}_n(-t^{-1}, qt^2, q)$$

In the genus-0 case $\mathcal{M}_n$ consists of a single point for $n = 1$ and has no points for $n > 1$. Hence $H(\mathcal{M}_n; q, t) = \delta_{n,1}$ which is consistent with the conjecture.

Theorem. (Rains–SOW, Carlsson–Rodriguez-Villegas (2016))

The conjecture holds for genus $g = 1$.

Proof.

The following q,t-analogue of the Nekrasov–Okounkov formula holds:

Theorem.

$$\sum_{\lambda} T^{|\lambda|} \prod_{s\in\lambda} \frac{(1-uq^{a(s)+1}t^{l(s)})(1-u^{-1}q^{a(s)}t^{l(s)+1})}{(1-q^{a(s)+1}t^{l(s)})(1-q^{a(s)}t^{l(s)+1})}$$
$$= \prod_{i,j,k\geqslant 1} \frac{(1-uq^{i}t^{j-1}T^{k})(1-u^{-1}q^{i-1}t^{j}T^{k})}{(1-q^{i-1}t^{j-1}T^{k})(1-q^{i}t^{j}T^{k})}$$

This may either be proved using Macdonald polynomial theory or the equivariant Dijkgraaf–Moore–Verlinde–Verlinde (DMVV) formula for the Hilbert scheme of n points in the plane, $(\mathbb{C}^2)^{[n]}$, due to Waelder.

Let (u_1, u_2) be the equivariant parameters of the natural torus action on $(\mathbb{C}^2)^{[n]}$, and set $q := \mathrm{e}^{-2\pi \mathrm{i} u_1}$ and $t := \mathrm{e}^{2\pi \mathrm{i} u_2}$. Let $\mathrm{Ell}\left((\mathbb{C}^2)^{[n]}; u, p, q, t\right)$ be the equivariant elliptic genus of $(\mathbb{C}^2)^{[n]}$, where $p := \exp(2\pi \mathrm{i}\tau)$ and $u := \exp(2\pi \mathrm{i} z)$ for $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$. According to the equivariant DMVV formula:

$$\sum_{n \geqslant 0} T^n \,\mathrm{Ell}\left((\mathbb{C}^2)^{[n]}; u, p, q, t\right) = \prod_{m \geqslant 0} \prod_{k \geqslant 1} \prod_{\ell, n_1, n_2 \in \mathbb{Z}} \frac{1}{(1 - p^m T^k u^\ell q^{n_1} t^{n_2})^{c(km, \ell, n_1, n_2)}}$$

The integers $c(m, \ell, n_1, n_2)$ are determined by the equivariant elliptic genus of $\mathbb{C}^2$ given by

$$\begin{aligned} \mathrm{Ell}(\mathbb{C}^2, u, p, q, t) &= \frac{\theta(uq; p)\theta(u^{-1}t; p)}{\theta(q; p)\theta(t; p)} \\ &= \sum_{m \geqslant 0} \sum_{\ell, n_1, n_2 \in \mathbb{Z}} c(m, \ell, n_1, n_2) p^m u^\ell q^{n_1} t^{n_2} \end{aligned}$$

where

$$\theta(u; p) := \sum_{k \in \mathbb{Z}} (-u)^k p^{\binom{k}{2}}$$

Li, Liu and Zhou obtained an explicit formula in terms of arm- and leg-lengths for the generating function (over n) of elliptic genera of the framed moduli spaces $M(r,n)$ of torsion-free sheaves on $\mathbb{P}^2$ of rank r and $c_2 = n$.

Since $M(1,n)$ coincides with $(\mathbb{C}^2)^{[n]}$ this implies

$$\sum_{n\geqslant 0} T^n \operatorname{Ell}\big((\mathbb{C}^2)^{[n]}; u,p,q,t\big) = \sum_{\lambda} T^{|\lambda|} \prod_{s\in\lambda} \frac{\theta(uq^{a(s)+1}t^{l(s)};p)\theta(u^{-1}q^{a(s)}t^{l(s)+1};p)}{\theta(q^{a(s)+1}t^{l(s)};p)\theta(q^{a(s)}t^{l(s)+1};p)}$$

This gives the elliptic Nekrasov–Okounkov formula

$$\sum_{\lambda} T^{|\lambda|} \prod_{s\in\lambda} \frac{\theta(uq^{a(s)+1}t^{l(s)};p)\theta(u^{-1}q^{a(s)}t^{l(s)+1};p)}{\theta(q^{a(s)+1}t^{l(s)};p)\theta(q^{a(s)}t^{l(s)+1};p)} = \prod_{m\geqslant 0}\prod_{k\geqslant 1}\prod_{\ell,n_1,n_2\in\mathbb{Z}} \frac{1}{(1-p^m T^k u^\ell q^{n_1} t^{n_2})^{c(km,\ell,n_1,n_2)}}$$