Metaplectic moments

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Outline

1. Exponentiated moment maps
   (a) Group-valued moment maps.
   (b) The metaplectic representation.
   - Weyl's commutation relations and the Moyal algebra.
   - Symplectic automorphisms.
   - Tempered implementors.
   - The connection with group-valued moment maps.

   (c) Reduced spaces

Born–Heisenberg commutation relations

$$[X_j, X_k] = 0, \qquad [P_j, P_k] = 0, \qquad [X_j, P_k] = i\hbar\delta_{jk}1$$

$$W(a, \alpha) := \exp(i(a \cdot P - \alpha \cdot X)/\hbar)$$

Weyl's commutation relations

$$W(a,\alpha)W(b,\beta) = e^{i(a\cdot\beta - b\cdot\alpha)/2\hbar}W(a+b,\alpha+\beta)$$

Generalisation to locally compact abelian groups V with symplectic form s and multiplier $\sigma = \exp(is(u,v)\hbar)$

$$W(u)W(v) = \sigma(u,v)W(u+v)$$

The twisted group algebra

For any Haar (Lebesgue) integrable function and, in particular, any Schwartz function $f \in \mathcal{S}(V)$ set

$$W(f) = \int_V f(v)W(v)\,dv.$$

Then

$$W(f_1)W(f_2) = W(f_1 * f_2),$$

where

$$(f_1 * f_2)(v) = \int_V f_1(u)f_2(v-u)\sigma(u, v-u)\,du$$

is twisted convolution.

The Clifford/fermion algebra is the twisted group algebra of $V = \mathbb{Z}_2^n$ with

$$\sigma(m,n) = (-1)^{\sum_{i>j} m_i n_j}.$$

The Moyal algebra

The classical situation has $\sigma = 1$ so that the multiplication is ordinary convolution and everything commutes. The analogue of $W(f)$ is the Fourier transform $f \mapsto \widehat{f}$, (which preserves $\mathcal{S}(V)$).

Weyl's quantisation $\mathcal{Q}(\widehat{f}) = W(f)$, sends the classical to the quantum transform.

$$\mathcal{Q}(\widehat{f_1 * f_2}) = W(f_1 * f_2) = W(f_1)W(f_2) = \mathcal{Q}(\widehat{f}_1)\mathcal{Q}(\widehat{f}_2)$$

We set

$$\widehat{f}_1 \star \widehat{f}_2 = \widehat{f_1 * f_2}$$

to get the Moyal product (a star product).

The star product expansion

Writing $\phi_j = \widehat{f_j}$, we see that $\phi_1 \star \phi_2$ is the Fourier transform of

$$\begin{aligned}
\phi_1 * \phi_2 &= \int_V e^{-is(u,v-u)/\hbar} f_1(u) f_2(v-u)\, du \\
&= \int_V f_1(u) f_2(v-u)\, du - \frac{i}{\hbar} \int_V s(u, v-u) f_1(u) f_2(v-u)\, du \\
&\quad - \frac{1}{2\hbar^2} \int_V s(u, v-u)^2 f_1(u) f_2(v-u)\, du + \ldots
\end{aligned}$$

The Poisson bracket

$$\phi_1 * \phi_2 = \int_V f_1(u) f_2(v-u)\, du - \frac{i}{\hbar}\int_V s(u, v-u) f_1(u) f_2(v-u)\, du + \dots$$

The first term is ordinary convolution and has the pointwise product $\phi_1\phi_2$ as Fourier transform. The multiplications by powers of $s(u, v-u)$ introduce derivatives $-\hbar^2 s(\partial_1, \partial_2)\phi_1\phi_2$ and the first two terms become

$$\phi_1 * \phi_2 = \phi_1\phi_2 + i\hbar s(\partial_1, \partial_2)\phi_1\phi_2 + \dots.$$

In particular, we get the Moyal commutator

$$\phi_1 * \phi_2 - \phi_2 = 2i\hbar s(\partial_1, \partial_2)\phi_1\phi_2 + \dots = i\hbar\{\phi_1, \phi_2\} + \dots,$$

the dominant term being the Poisson bracket.

Symplectic automorphisms

Since a symplectic transformation g of V preserves s and the addititive structure of V the action on a function f

$$(g.f)(v) = f(g^{-1}v)$$

preserves the twisted convolution.

Symplectic implementors

The symplectic automorphisms are not inner, but approximately inner:

There exist tempered distributions $S_g \in \mathcal{S}'(V)$ which are in the multiplier algebra, i.e. $S_g * \mathcal{S}(V) \subseteq \mathcal{S}(V)$ and $\mathcal{S}(V) * S_g \subseteq \mathcal{S}(V)$, such that

$$S_g * f = (g.f) * S_g$$

for all $f \in \mathcal{S}(V)$ and $g \in G = \mathrm{Sp}(V, s)$.

Algebra homomorphisms

Moreover these can be chosen such that

$$S_g * S_h = \alpha(g,h) S_{gh}$$

for all $g, h \in \mathrm{Sp}(V, s)$, where $\alpha(g,h) = \pm 1$ is a multiplier, which can be removed by going to the central extension which it defines, the metaplectic group, $\mathrm{Mp}(V, s)$, so they will be suppressed.

Explicit symplectic implementors

The implementation and homomorphism conditions determine S_g to within a multiplicative constant:

Theorem. (H 1981, *Math Proc. Camb. Phil. Soc.* **90** 465-476). The tempered distribution S_g is supported on $(g-1).V$ and given there by

$$S_g(v) = D^{-1} \det(g-1)^{-\frac{1}{2}} \exp\left[is\left(v, \left(\frac{g+1}{g-1}\right) v\right)/2\hbar\right],$$

where D is a constant.

(It is the choice of square root for $\det(g-1)$ which gives the multiplier.)

Equivariance

The distribution S_g is equivariant since

$$S_{hgh^{-1}} \star S_h = S_h \star S_g = (h \cdot S_g) \star S_h$$

from which it follows that

$$S_{hgh^{-1}} = (h \cdot S_g).$$

INVARIANTS

When h and g commute we have

$$S_g(h^{-1}v) = (h \cdot S_g)(v) = S_{hgh^{-1}}(v) = S_g(v).$$

Thus for any subgroup H, the distributions S_g with g in the centraliser H' of H are H invariant. In fact they are invariant under the centraliser H'' of H'.

Dual pairs

Two subgroups H and G such that $G = H'$ and $H = G'$ are said to be a dual pair.

Example.

$$H = \{h : (a,\alpha) \mapsto (\cos(\theta)a - \sin(\theta)\alpha, \cos(\theta)\alpha + \sin(\theta)a)\} \cong U(1)$$

has dual

$$H' = \{g : (a,\alpha) \mapsto (Aa - B\alpha, A\alpha + Ba) : A^\top A + B^\top B = 1, A^\top B = B^\top A\},$$

which is isomorphic to $U(n)$ since the conditions give $(A + iB)^*(A + iB) = 1$.

Howe duality

When one of a dual pair of subgroups is compact it is easy to prove that

the von Neumann algebra generated by $\{W(S_h) : h \in H\}$ is the commutant of the von Neumann algebra generated by $\{W(S_g) : g \in G\}$ and vice versa.

Howe's Duality Theorem says that this works for dual pairs of reductive subgroups.

The distributional approach suggests the conjecture that this generalises to other dual pairs.

The symplectic Fourier transform

Applying the general formula

$$S_g(v) = D^{-1} \det(g-1)^{-\frac{1}{2}} \exp\left[is\left(v, \left(\frac{g+1}{g-1}\right)v\right)/4\hbar\right],$$

we see that when $g = -1$, S_{-1} is a constant $C = D^{-1}\det(-2)^{-\frac{1}{2}}$, defined on the whole of V.

Its twisted convolution with any Schwartz function f is

$$(fS_{-1} * f)(v) = C\int_V f_2(v-u)\sigma(u, v-u)\,du = C\int_V f_2(w)e^{is(v-w,w)/\hbar}\,dw$$

so we could have used $S_{-1} * f$ as the Fourier transform of f, and henceforth will do so.

The Moyal implementors

The above Fourier transform is actually an involution, since
$f = S_{-1} * (S_{-1} * f)$.

With the above Fourier transform convention, and setting $\phi_j = S_{-1} * f_j$, we have

$$(\phi_1) \star (\phi_2) = S_{-1} * (f_1 * f_2) = \phi_1 * S_{-1} * \phi_2,$$

We now set $E_g = S_{-g} = S_{-1} * S_g = S_g * S_{-1}$, so that

$$E_g \star \phi = S_{-g} * S_{-1} * \phi = S_g * \phi = (g.\phi) * S_g = (g.\phi) * S_{-1} * E_g = (g.\phi) \star E_g.$$

Explicit Moyal implementors

Theorem (H 1981) The distribution E_g supported on $(g+1)\cdot V$ and given by

$$E_g(v) = D_\star^{-1}\det(g+1)^{-\frac{1}{2}}\exp\left(-i2s\left(v,\left(\frac{g-1}{g+1}\right)v\right)/\hbar\right).$$

satisfies

$$E_g \star \phi = (g.\phi)\star E_g, \qquad E_g \star E_h = E_{gh}, \qquad E_{hgh^{-1}} = h.E_g.$$

The infinitesimal moment map

The explicit formula shows that, for X in the Lie algebra $g = sp(V, s)$, $E_{\exp(tX)}$ can be differentiated with respect to t at $t = 0$, to give a function μ_X on $\mathbb{R}$. Differentiating the implementation relation gives

$$\mu_X \star f - f \star \mu_X = X.f$$

where X acts on f as the vector field derivation.

The definition of the Moyal product shows that $\mu_X \star f - f \star \mu_x = \{\mu_X, f\}$ is the Poisson bracket. so

$$\{\mu_X, f\} = X.f,$$

the moment-map property.

Relation to equivariant group-valued maps

Theorem (H 2011) Let $\mu : V \to G = \mathrm{Sp}(V, s)$ be continuous and equivariant in the sense that $g\mu(v)g^{-1} = \mu(g^{-1}v)$, and define $S_\mu(v) = S_{\mu(v)}(v)$. Then S_μ is a constant.

Sketch Proof

The orbits of $\mathrm{Sp}(V, s)$ on V are $\{0\}$ and $V \setminus \{0\}$. The equivariance property determines $S_m u$ on the larger orbit, and then continuity determines $S_\mu(0)$.

If we choose a non-zero vector $v_0 \in V$ and a cross-section $\gamma : V \to \mathrm{Sp}(V, s)$ such that $\gamma(v).v_0 = v$, then

$$\mu(v) = \mu(\gamma(v)v_0) = \gamma(v)\mu(v_0)\gamma(v)^{-1}$$

so that $\mu(v_0)$ determines the entire function.

Moreover,

$$S_{\mu(v)}(v) = S_{\gamma(v)\mu(v_0)\gamma(v)^{-1}}(\gamma(v).v_0) = \gamma(v).S_{\mu(v_0)}(\gamma(v).v_0) = S_{\mu(v_0)}(v_0)$$

independent of v.

$\mathrm{Sp}(V,s)$-valued moment maps

THEOREM (H 2011) Every equivariant $\mathsf{Sp}(V,s)$-valued map on V has the form $\mu_t^{\pm}(v) : w \mapsto \mu(w + ts(v,w)v)$ for some real t, and $\mu_t^{\pm}$ is an $\mathsf{Sp}(V,s)$-valued moment map.

Proof

The second part can be done by direct calculation: one just checks that $\mu_t^{\pm}$ is equivariant, commutes with the stabiliser of v, and satisfies the moment map conditions. These conditions involve checking that the canonical 3-form on $\mathsf{Sp}(V, s)$ pulls back to 0, (since s defines a closed symplectic form), and that the appropriate kernel is trivial since s is non-degenerate.

The first part exploits the equivariance requirement that $\mu(v)$ commutes with the stabiliser of v, which is a large enough parabolic subgroup to force μ to have the form $\mu_t^{\pm}$.

Homogeneous manifolds

Bowes and H (1997, *J. Geom. and Phys.* **22** 319–348):

Given

- compact group G,
- irreducible representation D on a finite-dim. inner product space $\mathcal{H}_D$,
- set $\mathrm{ad}_D(x)[\rho = D(x)\rho D(x)^{-1}$,
- positive self-adjoint operator ρ on $\mathcal{H}_D$, with $\mathrm{tr}[\rho] = 1$,
- K be the subgroup of $k \in G$ such that $D(k)$ commutes with ρ.

we have $\mathrm{ad}_D(xk)[\rho = D(xk)\rho D(xk)^{-1}$ is independent of k.

Homogeneous manifolds

Using the trace (Hilbert-Schmidt) inner product on operators, for each operator A on $\mathcal{H}_D$ define

$$f_A(x) = \langle \mathrm{ad}_D(x)[\rho], A\rangle_{\mathrm{tr}}.$$

Since $f_A(x)$ depends only on the coset xK.it gives a function on the homogeneous space $M = G/K$.

A star product

For suitable ρ and D the map $A \mapsto f_A(\cdot) = \langle \mathrm{ad}_D(\cdot)[\rho], A\rangle_{\mathrm{tr}}$ is invertible and gives a quantisation of C(G/K). Then we can define a star product on functions by

$$f_A \star f_B = f_{AB}$$

When G can be imbedded in a symplectic group so that it is the centraliser of its centraliser then this structure can be obtained by symplectic reduction of the Moyal product.

Borel–Weil–Bott

When G is a compact Lie group, Ω a highest weight vector and ρ projection onto Ω we get an explicit realisation of $\mathcal{H}_D$ as holomorphic sections of a line bundle $\mathcal{L}$ over G/K. Tensor powers $\Omega^{\otimes k}$ are associated with $\mathcal{L}^k$, For A on the symmetric tensor power $\otimes_S^r \mathcal{H}_D$ one has the natural injection $A \otimes_S 1^{\otimes(k-r)}$ on $\otimes^k \mathcal{H}_D$ one can identify f_A corresponding to different spaces, and we write $\star_k$ for the star product (so that $\star_1 = \star$).

The star product expansion

Intuitively k behaves like $1/\hbar$.

Theorem (Bowes and H 1997) For A on $\otimes_S^r \mathcal{H}_D$ and B on $\otimes_S^s \mathcal{H}_D$ one has an expansion of the form

$$f_A \star f_B = \sum_{p=\max(0,r+s-k)}^{\min(r,s)} \frac{(k-r)!(k-s)!}{k!(k+p-r-s)!} f_A \circ_p f_B$$

where $\circ_0$ is the pointwise product, and $\circ_p$ are other explicitly defined products.

The star product expansion

In the classical limit of large k the general formula gives

$$f_A \star f_B \quad \sim \sum_{p=\max(0,r+s-k)}^{\min(r,s)} k^{-p} f_A \circ_p f_B \quad \sim \quad f_A f_B$$

In the particular case when $r = s = 1$ the expansion is

$$f_A \star_k f_B = f_A f_B + \frac{1}{k}(f_{AB} - f_A f_B),$$

where $f_A f_B$ is the pointwise product.

Frønsdal exponential maps

We define exponential functions by

$$E_g(xK) = f_{D(g)}(xK) = \mathrm{tr}[\rho D(x^{-1}gx)]$$

This ensures that

$$E_g \star f_A = f_{D(g)} \star f_A = f_{D(g)A}.$$

In particular when $A = D(h)$ this gives

$$E_g \star E_h = f_{D(gh)} = E_{gh},$$

and also

$$E_g \star f_A = f_{\mathrm{ad}_D(g)A} \star E_g.$$

Star product implementors

Now

$$\begin{aligned} f_{\mathrm{ad}_D(g)A}(x) &= \langle \mathrm{ad}_D(x)\rho, \mathrm{ad}_D(g)A\rangle_{\mathrm{tr}} \\ &= \langle \mathrm{ad}_D(g^{-1}x)\rho, A\rangle_{\mathrm{tr}} \\ &= f_A(g^{-1}x) = (g.f_A)(x), \end{aligned}$$

so that

$$E_g \star f_A = (g.f_A) \star E_g.$$

Moment maps

As in the metaplectic case, setting $\mu_Y = dE_{\exp(tY)}/dt|_{t=0}$ gives

$$\mu_Y \star f_A - f_A \star \mu_Y = v_Y.f_A$$

where v_Y is the vector field defined by $Y \in g$.

Commutators with respect to the star product are connected to the standard symplectic form, as we see by defining the element $\xi_\rho \in g^*$ by

$$\xi_\rho(Y) = \langle \rho, \dot{D}(Y) \rangle$$

where $\dot{D}$ denotes the representation of g obtained from D. This gives

$$[f_{\dot{D}(Y)}, f_{\dot{D}(Z)}]_\star = f_{[\dot{D}(Y),\dot{D}(Z)]} = \langle \mathrm{ad}_D(x)\rho, \dot{D}[Y,Z] \rangle = (\mathrm{ad}^*(x)\xi_\rho)([Y,Z]).$$

THE END