A higher supergroup for string theory

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This research began as a puzzle. Explain this pattern:

- The only normed division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. They have dimensions $k = 1, 2, 4$ and 8.
- The classical superstring makes sense only in dimensions $k + 2 = 3, 4, 6$ and 10.
- The classical super-2-brane makes sense only in dimensions $k + 3 = 4, 5, 7$ and 11.

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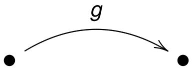
The explanation involves 'higher gauge theory'.

Ordinary gauge theory describes how 0-dimensional particles transform as we move them along 1-dimensional paths. It is natural to assign a Lie group element to each path:



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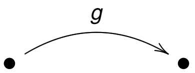
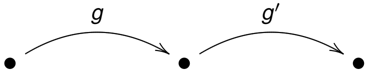


since composition of paths then corresponds to multiplication:



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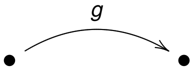

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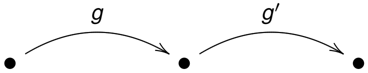
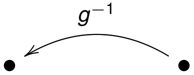
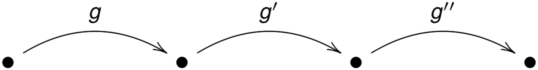


while reversing the direction corresponds to taking the inverse:



The associative law makes the holonomy along a triple composite unambiguous:



So: *the topology dictates the algebra!*

Higher gauge theory describes the parallel transport not only of point particles, but also 1-dimensional strings.

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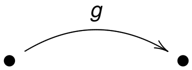
A ‘2-group’ has objects:



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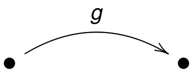
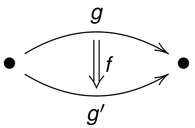


but also morphisms:



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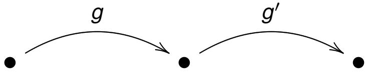

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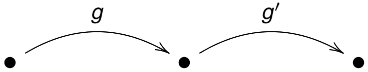

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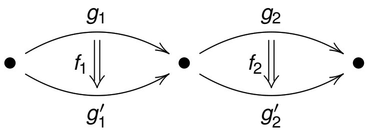

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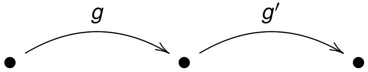

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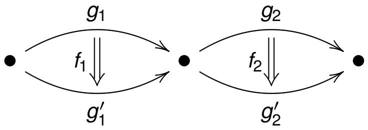
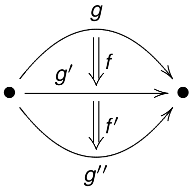


and also compose morphisms:



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To study *superstrings* using higher gauge theory, we really want ‘Lie 2-*supergroups*’.

But to get our hands on these, it’s easiest to start with ‘Lie 2-superalgebras’.

Let's really start with 'Lie 2-algebras'.

Roughly, this is a categorified Lie algebra: a category L equipped with a functor:

$$[-,-]\colon L \times L \to L,$$

where the Lie algebra axioms only hold up to isomorphism:

Axiom	**Lie algebra**	**Lie 2-algebra**
Jacobi identity	$[x,[y,z]] + \text{cyclic} = 0$	$[x,[y,z]] + \text{cyclic} \cong 0$

(We'll not weaken antisymmetry $[x,y] = -[y,x]$: this is called a **semistrict** Lie 2-algebra.)

A Lie 2-algebra contains a lot of information: objects, morphisms, and the isomorphisms weakening the Lie algebra axioms.

Fortunately, we can distill a Lie 2-algebra down to only four pieces of data:

Theorem (Baez–Crans)

Up to equivalence, every semistrict Lie 2-algebra is determined by the quadruple $(\mathfrak{g}, \mathfrak{h}, \rho, J)$, where:

- $\mathfrak{g}$ is a Lie algebra,
- $\mathfrak{h}$ is a vector space,
- $\rho \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$ is a representation of $\mathfrak{g}$ on $\mathfrak{h}$,
- $J \colon \Lambda^3 \mathfrak{g} \to \mathfrak{h}$ is a 3-cocycle in Lie algebra cohomology.

In short: give me a Lie algebra 3-cocycle, and I'll give you a Lie 2-algebra, and vice versa.

'3-cocycle' in the sense of **Lie algebra cohomology**, which is defined using the complex which consists of antisymmetric *p*-linear maps at level *p*:

$$C^p(\mathfrak{g}, \mathfrak{h}) = \{\omega \colon \Lambda^p \mathfrak{g} \to \mathfrak{h}\}$$

We call ω a **Lie algebra *p*-cochain.**

This complex has coboundary map $d \colon C^p(\mathfrak{g}, \mathfrak{h}) \to C^{p+1}(\mathfrak{g}, \mathfrak{h})$ defined by a long formula.

When $d\omega = 0$, we say ω is a **Lie algebra *p*-cocycle**.

Remember, we really want Lie 2-superalgebras:

- $\mathfrak{g}$ is a Lie superalgebra:
  - a super vector space:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

  - with a graded-antisymmetric bracket $[-,-]$, satisying the Jacobi identity up to some signs.
- $\mathfrak{h}$ is a super vector space: $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$.
- $\rho \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$ is a homomorphism of Lie superalgebras.
- $J \colon \Lambda^3 \mathfrak{g} \to \mathfrak{h}$ is a Lie superalgebra 3-cocycle.

In short: give me a Lie superalgebra 3-cocycle, and I'll give you a Lie 2-superalgebra.

The **Poincaré superalgebra**:

- $V = \mathbb{R}^{n-1,1}$ has a nondegerate bilinear form g.
- **Spinor representations** of $\mathfrak{so}(n-1,1)$ are representations arising from left-modules of $\mathrm{Cliff}(V) = \frac{TV}{vw+wv=2g(v,w)}$, since

$$\mathfrak{so}(n-1,1) \hookrightarrow \mathrm{Cliff}(V).$$

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- Let S be such a representation.
- For the right choice of S, there is a symmetric map:

$$[-,-] \colon \mathrm{Sym}^2 S \to V.$$

- There is thus a Lie superalgebra $\mathfrak{siso}(n-1,1)$ where:

$$\mathfrak{siso}(n-1,1)_0 = \mathfrak{so}(n-1,1) \ltimes V, \quad \mathfrak{siso}(n-1,1)_1 = S.$$

Theorem

For spacetimes of dimension $n = 3, 4, 6$ and 10, there is a Lie 2-superalgebra $\mathfrak{superstring}(n-1,1)$ defined by the quadruple $(\mathfrak{g}, \mathfrak{h}, \rho, J)$ where:

- $\mathfrak{g} = \mathfrak{siso}(n-1,1)$ is the Poincaré superalgebra—the infinitesimal symmetries of 'superspacetime'.
- $\mathfrak{h} = \mathbb{R}$,
- the action ρ is trivial,
- the 3-cocycle J is zero except for:

$$J(v, \psi, \phi) = g(v, [\psi, \phi]),$$

for a translation v and two 'supertranslations' ψ and ϕ.

Why $\mathfrak{superstring}(n-1,1)$?

- The classical superstring only makes sense in dimensions $n = 3$, 4, 6 and 10.
- In the physics literature, we see this is because J is a cocycle *only in these dimensions*. We can explain this using division algebras.
- Sati–Schreiber–Stasheff have a theory of connections valued in Lie 2-algebras. With $\mathfrak{superstring}(n-1,1)$, the background fields look right.

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With some work, we can integrate $\mathfrak{superstring}(n-1,1)$ to a '2-group'.

A **2-group** is a category $\mathcal{G}$ with invertible morphisms, equipped with a functor called **multiplication**:

$$m \colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}.$$

The presence of isomorphisms allow us to *weaken the group axioms*:

Axiom	Group	2-group
Associativity	$(xy)z = x(yz)$	$(xy)z \cong x(yz)$
Left and right units	$1x = x = x1$	$1x \cong x \cong x1$
Inverses	$xx^{-1} = 1 = x^{-1}x$	$xx^{-1} \cong 1 \cong x^{-1}x$

These isomorphisms then must satisfy some equations of their own.

The analogue of Baez and Crans' theorem holds:

Theorem (Joyal–Street)

Up to equivalence, every 2-group is determined by the quadruple (G, H, α, a), where:

- G is a group,
- H is an abelian group,
- $\alpha \colon G \to \mathrm{Aut}(H)$ gives an action of G on H,
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In short: give me a group 3-cocycle, and I'll give you a 2-group, and vice versa.

'3-cocycle' in the sense of **group cohomology**, which is defined using the complex of G-equivariant maps:

$$C^p(G, H) = \left\{ F \colon G^{p+1} \to H \mid gF(g_0, \dots, g_p) = F(gg_0, \dots, gg_p) \right\}$$

We call F a **group p-cochain.** This complex has coboundary map $d \colon C^p(G, H) \to C^{p+1}(G, H)$ defined by:

$$dF(g_0, \dots, g_p, g_{p+1}) = \sum_{i=0}^{p+1} (-1)^i F(g_0, \dots, \hat{g}_i, \dots, g_p, g_{p+1}).$$

When $dF = 0$, we say F is a **group p-cocycle.**

For physics, we really need 'Lie 2-groups':

- G and H are Lie groups, α and a are smooth maps.

For the physics of superstrings, we really need 'Lie 2-supergroups':

- G and H are 'Lie supergroups', α and a are 'super smooth'.

In short:

- Give me a smooth 3-cocycle $a\colon G^4 \to H$, and I'll give you a Lie 2-group.
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- Give me a super smooth 3-cocycle $a\colon G^4 \to H$, and I’ll give you a Lie 2-supergroup.
- But not vice versa: I haven’t defined Lie 2-(super)groups in general here, I am just using 3-cocycles as a substitute.

So we have a Lie 2-superalgebra, $\mathfrak{superstring}(n-1,1)$. But we want a 2-supergroup. It's easy to go the other way:

- $\mathfrak{g}$ is the Lie superalgebra of G,
- $\mathfrak{h}$ is the Lie superalgebra of H,
- $d\alpha\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$ gives the induced action of $\mathfrak{g}$ on $\mathfrak{h}$,
- $Da\colon \Lambda^3\mathfrak{g} \to \mathfrak{h}$ comes from differentiating $a\colon G^4 \to H$ at 1, and antisymmetrizing.

Da is still a cocycle. In general, there is a cochain map:

$$D\colon C^p(G,H) \to C^p(\mathfrak{g},\mathfrak{h}).$$

We want to solve the inverse problem. Our 3-cocycle formalism suggests a way to integrate Lie 2-superalgebras to Lie 2-supergroups:

- integrate $\mathfrak{g}$ to G,
- integrate $\mathfrak{h}$ to H,
- integrate the action ρ to an action α of G on H,
- find a cocycle $a \colon G^4 \to H$ which somehow integrates $J \colon \Lambda^3\mathfrak{g} \to \mathfrak{h}$.

More precisely, we want a cochain map:

$$\int \colon C^p(\mathfrak{g}, \mathfrak{h}) \to C^p(G, H)$$

which is the inverse of differentiation, at least up to chain homotopy:

$$D \colon C^p(G, H) \to C^p(\mathfrak{g}, \mathfrak{h}).$$

It's not always possible to integrate cocycles, but it is for the defining cocycle J on the $\mathfrak{superstring}(n-1,1)$ Lie 2-superalgebra.

This is because J is supported on the Lie subalgebra $\mathcal{T} = V \oplus S$, a *nilpotent Lie superalgebra* of translations and supertranslations.

As a warm up, we show how to integrate $\mathbb{R}$-valued cocycles on any nilpotent Lie *algebra* $\mathfrak{n}$ to cocycles on the corresponding group N, using a technique due to Houard.

In this case $\exp\colon \mathfrak{n} \to N$ is a diffeomorphism, and gives a notion of straight lines in N.

- Lie algebra cochains $\omega \colon \Lambda^p \mathfrak{n} \to \mathbb{R}$ can be identified with left-invariant differential forms on N.
- We can define **left-invariant simplices** in N to be simplices:
$$[n_0, \dots, n_p] \colon \Delta^p \to N,$$
with the property:
$$n[n_0, \dots, n_p] = [nn_0, \dots, nn_p].$$
- We integrate to get Lie group cochains on N:
$$\textstyle\int \omega(n_0, \dots, n_p) = \displaystyle\int_{[n_0, \dots, n_p]} \omega.$$
- This defines a cochain map by Stokes' theorem!

Now, we move into the world of supermanifolds:

Using a beautiful trick, called the **functor of points.**

Theorem (Balduzzi–Carmeli–Fioresi)

There is a full and faithful functor:

$$h \colon \mathrm{SuperManifolds} \to \mathrm{Fun}(\mathrm{GrassmannAlg}, \mathcal{A}_0\text{-Manifolds}).$$

So: for any supermanifold M and Grassmann algebra A, we get a *manifold* $h(M)(A) = M_A$, the ***A*-points** of M.

For any map $f \colon M \to N$ of supermanifolds, we get a smooth map $f_A \colon M_A \to N_A$, which defines a natural transformation.

The functor of points allows us to replace a supermanifold M with a family of ordinary manifolds M_A, and maps $f \colon M \to N$ with a family of smooth maps $f_A \colon M_A \to N_A$.

The A-points of a super vector space V are:

$$V_A = A_0 \otimes V_0 \,\oplus\, A_1 \otimes V_1 .$$

The A-points of $\mathcal{T}$ are $\mathcal{T}_A = A_0 \otimes \mathcal{T}_0 \oplus A_1 \otimes \mathcal{T}_1$.

- $\mathcal{T}$ a nilpotent Lie superalgebra $\Rightarrow \mathcal{T}_A$ a nilpotent Lie algebra.
- J a 3-cocycle on $\mathcal{T} \Rightarrow J_A$ a 3-cocycle on $\mathcal{T}_A$.
- $\mathcal{T}_A$ has a group structure, $T_A \Rightarrow \mathcal{T}$ has a supergroup structure T.

So: we integrate to get $\int J_A$ and transfer this back to T, defining $\int J$ on T.

Theorem

$\int J$ defines a Lie supergroup 3-cocycle on T, which extends to a Lie supergroup 3-cocycle on $\mathrm{SISO}(n-1,1)$.

Corollary

There is a Lie 2-supergroup $\mathrm{Superstring}(n-1,1)$ integrating $\mathfrak{superstring}(n-1,1)$.

Final thoughts:

- We want to do ‘higher gauge theory’ with $\mathrm{Superstring}(n-1,1)$. This should be related to string theory, and to the work of Sati–Schreiber–Stasheff.
- There is also a Lie 3-supergroup $2\text{-}\mathrm{Brane}(n,1)$ associated with super-2-branes. The higher gauge theory should be related to M-theory.

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- This talk is based on a paper:

  Division algebras and supersymmetry III

  coming soon to an *n*-Category Café near you!