Quantum group actions on rings and equivariant algebraic K-theory

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Joint work with Gus Lehrer

Motivations

$\mathrm{U} = \mathrm{U}_q(\mathfrak{g})$, a quantum group,
A, a *U*-module algebra over U, that is, an associative $\Bbbk$-algebra with a U-action that preserves the algebraic structure of A.

The subspace A^{U} of U-invariants forms a subalgebra.

Describe the structure of the subalgebra A^{U} of invariants: a quantum analogue of the first fundamental theorem of invariant theory for the quantum groups associated with classical Lie algebras was established (joint work with Gus Leher and Hechun Zhang [LZZ11]).

"Higher invariants" of U-module algebras?

• U-module algebra $A \leftrightsquigarrow$ noncommutative space X with U-action;

• finitely generated projective U-equivariant A-module $\leftrightsquigarrow$ equivariant noncommutative vector bundle on X;

K-groups of such quivariant noncommutative vector bundles are invariants of A.

Equivariant modules

Let $\mathfrak{g}$ be a finite dim'l complex simple Lie algebra.
Let $\mathrm{U}_q(\mathfrak{g})$ be the quantum group associated with $\mathfrak{g}$ defined over $\mathbb{k} := \mathbb{C}(q)$, the field of rational functions in q.
Then U has the structure of a Hopf algebra.

Example: $\mathrm{U}_q(sl_2)$ is generated by $e, f, k^{\pm 1}$ with relations

$$kk^{-1} = k^{-1}k = 1,$$
$$kek^{-1} = q^2 e, \quad kfk^{-1} = q^{-2} f,$$
$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

The co-multiplication is given by

$$\Delta(k) = k \otimes k,$$
$$\Delta(e) = e \otimes k + 1 \otimes e,$$
$$\Delta(f) = f \otimes 1 + k^{-1} \otimes f.$$

Back to an arbitrary quantum group $\mathrm{U} = \mathrm{U}_q(\mathfrak{g})$.
For any $x \in \mathrm{U}$, write co-multiplication $\Delta(x)$ as

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}.$$

An associative algebra A is a module algebra over U if A is a U-module, and the algebraic structure is preserved by the action in the sense that, for all $a, b \in A$ and $x \in \mathrm{U}$,

$$x \cdot (ab) = \sum_{(x)} (x_{(1)} \cdot a)(x_{(2)} \cdot b),$$
$$x \cdot 1 = \epsilon(x) 1.$$

Here $\epsilon : \mathrm{U} \longrightarrow \mathbb{k}$ is the counit.

A U-equivariant A-module (or A-U-module) M is

- an A-module, $\phi : A \otimes M \longrightarrow M$,
- a U-module, $\mu : \mathrm{U} \otimes M \longrightarrow M$,
- the two module structures are compatible in the sense that the following diagram commutes

$$\begin{array}{ccc} \mathrm{U} \otimes (A \otimes M) & \stackrel{\mathrm{id}\otimes\phi}{\longrightarrow} & \mathrm{U} \otimes M \\ \mu' \downarrow & & \mu \downarrow \\ A \otimes M & \stackrel{\phi}{\longrightarrow} & M, \end{array}$$

where μ' is the U-module structure map of $A \otimes_{\Bbbk} M$

$$x \otimes (a \otimes m) \quad \mapsto \quad \sum_{(x)} x_{(1)} \cdot a \otimes x_{(2)} m$$

A morphism between two A-U-modules is a map which is both A-linear and U-linear.

- A-U-**mod**, the category of locally U-finite A-U-modules. [A U-module V is locally finite if $\dim \mathrm{U}v < \infty$ for any $v \in V$. Locally finite U-modules are semi-simple.]
- $\mathcal{M}(A, \mathrm{U})$, the full subcategory of A-U-**mod** consisting of finitely A-generated objects.
- $\mathcal{P}(A, \mathrm{U})$, the full subcategory of A-U-**mod** consisting of finitely generated projective objects.

Hereafter we fix a *locally finite* U-module algebra A.

Set $V_A = A \otimes_{\Bbbk} V$ for any finite dimensional U-module V.
Let A act on V_A by left multiplication, and
let U act by

$$x(a \otimes v) = \sum_{(x)} x_{(1)} \cdot a \otimes x_{(2)} v$$

for all $a \in A$, $v \in V$ and $x \in \mathrm{U}$.

Call V_A a free A-U-module of finite rank.
Facts:

1. V_A belong to $\mathcal{M}(A, \mathrm{U})$.
2. For each object M of $\mathcal{M}(A, \mathrm{U})$, there exists a V_A and surjection $V_A \to M$ in $\mathcal{M}(A, \mathrm{U})$.

Choose any finite set of A-generators for M in $\mathcal{M}(A, \mathrm{U})$. The U-module V generated by the set must be finite dimensional because of the local U-finiteness of M. Let $V_A = A \otimes_{\Bbbk} V$. Then the A-U-map $V_A \longrightarrow M$, $a \otimes v \mapsto av$, is surjective.

Splitting Lemma

Let $0 \longrightarrow M' \longrightarrow M \xrightarrow{p} M'' \longrightarrow 0$ be a short exact sequence in A-U-**mod** where M'' is an object of $\mathcal{M}(A, U)$. If the exact sequence is A-split, then it is also split as an exact sequence of A-U-modules.

Proof.

Facts: For any A-U-modules W and N, there is a U-action on $\mathrm{Hom}_A(W, N)$ defined for any $x \in \mathrm{U}$ and $f \in \mathrm{Hom}_A(W, N)$ by

$$(xf)(m) = \sum_{(x)} x_{(2)} f(S^{-1}(x_{(1)})m), \quad \forall m \in W.$$

If $W \in \mathcal{M}(A, \mathrm{U})$, and N is locally U-finite, then $\mathrm{Hom}_A(W, N)$ is a semi-simple U-module.

The sequence $\mathrm{Hom}_A(M'', M) \xrightarrow{p\circ -} \mathrm{Hom}_A(M'', M'') \longrightarrow 0$ is exact, and $p \circ -$ is a U-map. Both U-modules in the sequence are semi-simple,

$$\mathrm{Hom}_{A-\mathrm{U}}(M'', M) \xrightarrow{p\circ -} \mathrm{Hom}_{A-\mathrm{U}}(M'', M'') \longrightarrow 0$$

is exact. Thus any pre-image of $\mathrm{id}_{M''}$ splits the original exact sequence of A-U-modules.

Corollary

The following conditions are equivalent for an object P *of* $\mathcal{M}(A, \mathrm{U})$*:*

1. P *is projective as an* A*-module;*
2. P *is a projective object of* A-U-**mod***;*
3. P *is a direct summand of* $V_A := A \otimes_{\Bbbk} V$ *with* $\dim V < \infty$.

Some notions from Quillen's K-theory

Theorem

$\mathcal{P}(A, \mathrm{U})$ is an exact category.

Thus Quillen's K-groups $K_i(\mathcal{P}(A, \mathrm{U}))$ are defined.

Definition

Let $K_i^{\mathrm{U}}(A) := K_i(\mathcal{P}(A, \mathrm{U}))$ for $i = 0, 1, \ldots,$ and call them the U-equivariant algebraic K-groups of the U-module algebra A.

Exact category.

An exact category $\mathcal{M}$ is an additive category with a class **E** of short exact sequences which satisfies a series of axioms.

May think of an exact category $\mathcal{M}$ as a full (additive) subcategory of an abelian category $\mathcal{A}$ which is closed under extensions in $\mathcal{A}$, that is, if $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is exact in $\mathcal{A}$ and M' and M'' are in $\mathcal{M}$, then M also belongs to $\mathcal{M}$.

Typical examples of exact categories:
(1). any abelian category with the exact structure given by all the short exact sequences;
(2). the full subcategory of finitely generated projective (left) modules of the category of (left) modules over a ring.

A functor $F : \mathcal{M} \longrightarrow \mathcal{M}'$ between two exact categories $(\mathcal{M}, \mathbf{E})$ and $(\mathcal{M}', \mathbf{E}')$ is called exact if $F(\mathbf{E}) \subset \mathbf{E}'$.

Quillen categories of exact categories.

For a small exact category $\mathcal{M}$, the Quillen category $\mathcal{QM}$ has the same objects as $\mathcal{M}$, but with morphisms defined in the following way. Let M and M' be objects in $\mathcal{M}$ and consider all diagrams

$$M \overset{j}{\twoheadleftarrow} N \overset{i}{\rightarrowtail} M',$$

where j is an admissible epimorphism and i an admissible monomorphism. An admissible monomorphism (resp. admissible epimorphism) in $\mathcal{M}$ is a map that occurs as the map i (resp. j) in some member $0 \longrightarrow M_1 \overset{i}{\longrightarrow} M_0 \overset{j}{\longrightarrow} M_2 \longrightarrow 0$ of $\mathbf{E}$.

Two such diagrams are regarded as equivalent if there exists a commutative diagram

$$\begin{array}{ccccc} M & \longleftarrow & N & \longrightarrow & M' \\ \| & & \downarrow\cong & & \| \\ M & \longleftarrow & \tilde{N} & \longrightarrow & M'. \end{array}$$

A morphism from M to M' in the category $\mathcal{QM}$ is by definition an equivalence class of these diagrams.

Given a morphism from M' to M'' represented by the diagram

$$M' \overset{j'}{\twoheadleftarrow} N' \overset{i'}{\rightarrowtail} M'',$$

its composition with the morphism from M to M' is the morphism represented by

$$M \overset{j \circ p_1}{\twoheadleftarrow} N \times_{M'} N' \overset{i' \circ p_2}{\rightarrowtail} M'',$$

where $N \times_{M'} N' = \{(n, n') \in N \times N' \mid i(n) = j'(n')\}$ is the fibre product of N and N' over M', and $p_1 : N \times_{M'} N \twoheadrightarrow N$ and $p_2 : N \times_{M'} N \twoheadrightarrow N'$ are the obvious projections.
The Quillen category can be defined for any exact category of which the isomorphism class of objects form a set.

Classifying space of a category.

The classifying space $B(\mathcal{M})$ of a category $\mathcal{M}$ with a set of isomorphism classes of objects is a CW complex whose p-cells are in one to one correspondence with sequences in $\mathcal{M}$ of the form

$$X_0 \longrightarrow X_1 \longrightarrow \ldots \longrightarrow X_p$$

such that none of the maps is an identity map. The p-cell associated with the above sequence is attached in the obvious way to any cell of smaller dimension that can be obtained by deleting some X_i, and replacing f_i and f_{i+1} by $f_{i+1} \circ f_i$ if $i \neq 0$ or n. [One cancels it if the composition of maps leads to an identity.]

Quillen's algebraic K-theory of exact categories.

Quillen's algebraic K-groups of an exact category $\mathcal{M}$ are defined to be the homotopy groups of the classifying space $B(\mathcal{QM})$ of $\mathcal{QM}$:

$$K_i(\mathcal{M}) = \pi_{i+1}(B(\mathcal{QM})), \quad i = 0, 1, \ldots.$$

If $F : \mathcal{M} \longrightarrow \mathcal{N}$ is an exact functor between exact categories, it induces a functor $\mathcal{Q}F : \mathcal{QM} \longrightarrow \mathcal{QN}$ between the corresponding Quillen categories. This functor then induces a cellular map $B\mathcal{Q}F : B(\mathcal{QM}) \longrightarrow B(\mathcal{QN})$, which in turn leads to homomorphisms of K-groups

$$F_* : K_i(\mathcal{M}) \longrightarrow K_i(\mathcal{N}), \quad \text{for all } i.$$

Regular algebras

Recall that a left Noetherian algebra A is left regular if every finitely generated left A-module has a finite resolution by finitely generated projective A-modules.

Theorem

Assume that the U*-module algebra* A *is left regular. Then every object* M *in* $\mathcal{M}(A, \mathrm{U})$ *admits a finite* $\mathcal{P}(A, \mathrm{U})$*-resolution.*

Proof.

For any object M in $\mathcal{M}(A, \mathrm{U})$, there exists an exact sequence $V_{0,A} \xrightarrow{p_0} M \longrightarrow 0$ in $\mathcal{M}(A, \mathrm{U})$, where $V_{0,A}$ is a free A-U-module. As A is left Noetherian, $ker(p_0)$ belongs to $\mathcal{M}(A, \mathrm{U})$. Same considerations apply, leading inductively to an A-free resolution $\ldots \longrightarrow V_{1,A} \longrightarrow V_{0,A} \longrightarrow M \longrightarrow 0$ in $\mathcal{M}(A, \mathrm{U})$.

Let d be the projective dimension of M, which is finite because A is regular. The kernel P of the map $V_{d-1,A} \longrightarrow V_{d-2,A}$ is A-projective, hence belongs to $\mathcal{P}(A, \mathrm{U})$. Thus we arrive at the $\mathcal{P}(A, \mathrm{U})$-resolution

$$0 \longrightarrow P \longrightarrow V_{d-1,A} \longrightarrow \ldots \longrightarrow V_{1,A} \longrightarrow V_{0,A} \longrightarrow M \longrightarrow 0.$$

□

Theorem

If the U*-module algebra* A *is left regular, there exist isomorphisms*

$$K_i^{\mathrm{U}}(A) \xrightarrow{\sim} K_i(\mathcal{M}(A, \mathrm{U})), \quad i = 0, 1, 2, \ldots.$$

Proof.

Since A is left regular, it must be left Noetherian. Thus $\mathcal{M}(A, \mathrm{U})$ is an abelian category, which has the natural exact structure consisting of all the short exact sequences. Thus $K_i(\mathcal{M}(A, \mathrm{U}))$ are defined.

The embedding $\mathcal{P}(A, \mathrm{U}) \subset \mathcal{M}(A, \mathrm{U})$ satisfies the conditions of Quillen's Resolution Theorem, thus the claim immediately follows. □

Filtered module algebras

Let S be a locally finite U-module algebra with a filtration

$$0 = F_{-1}S \subset F_0 S \subset F_1 S \subset \ldots,$$

where $1 \in F_0 S$, $\cup_p F_p S = S$ and $F_p S \cdot F_q S \subset F_{p+q} S$.
Assume that the filtration is preserved by the U-action.
Let $\quad \overline{S} = gr(S), \quad \overline{S}_+ = \oplus_{p>0} \overline{S}_p, \quad A = F_0 S.$

Theorem

Assume that $\overline{S}$ is left Noetherian and A-flat. If A $(= \overline{S}/\overline{S}_+)$ has a finite projective $\overline{S}$-resolution, then there exist the isomorphisms

$$K_i(\mathcal{M}(A, \mathrm{U})) \xrightarrow{\sim} K_i(\mathcal{M}(S, \mathrm{U})), \quad \forall i = 0, 1, 2, \ldots.$$

If furthermore A is regular, then S is regular, and there exist the isomorphisms $\quad K_i^{\mathrm{U}}(A) \xrightarrow{\sim} K_i^{\mathrm{U}}(S), \quad i = 0, 1, 2, \ldots.$

The proof for the first part is involved, but that for the second part follows from previous theorem.

Quantum symmetric algebras

V, a finite dim'l U-module.
$T(V)$, tensor algebra of V.
$I \subset V \otimes_{\Bbbk} V$, U-submodule.
$\langle I \rangle$, two-sided ideal of $T(V)$ generated by I.
Then $\Bbbk\{V, I\} := T(V)/\langle I \rangle$ is a U-module algebra.

Call $A = \Bbbk\{V, I\}$ a quantum symmetric algebra of the U-module V if it has a PBW basis [that is, there exists some basis $\{v_i \mid i = 1, 2, \ldots, d\}$ of V such that the elements $v^{\mathbf{a}} := v_1^{a_1} v_2^{a_2} \cdots v_d^{a_d}$, with $\mathbf{a} := (a_1, a_2, \ldots, a_d) \in \mathbb{Z}_+^d$, form a basis of A.].

Theorem

Assume that the quantum symmetric algebra $A = \Bbbk\{V, I\}$ is left Noetherian. Then A is regular, and

$$K_i^{\mathrm{U}}(A) = K_i(\mathrm{U}\text{-}\mathbf{mod}) \quad \textit{for all } i \geq 0,$$

*were U-**mod** is the category of finite dim'l left U-modules.*

A is graded with $A_0 = \mathbb{k}$. If we can show that A satisfies the conditions of the theorem on equivariant K-groups of filtered algebras, then

$$K_i^{\mathrm{U}}(A) = K_i^{\mathrm{U}}(\mathbb{k}).$$

We have $K_i^{\mathrm{U}}(\mathbb{k}) = K_i(\mathcal{P}(\mathbb{k}, \mathrm{U}))$.

Note that $\mathcal{M}(\mathbb{k}, \mathrm{U})$ is the category of finite dimensional left U-modules, that is U-**mod**. As $\mathcal{M}(\mathbb{k}, \mathrm{U})$ is semi-simple, we have $\mathcal{P}(\mathbb{k}, \mathrm{U}) = \mathcal{M}(\mathbb{k}, \mathrm{U})$.

Thus we only need to show that $A_0 = \mathbb{k}$ has a finite projective A-resolution.

A quantum symmetric algebra is Koszul as it has a PBW basis [P] by definition. The generalised Koszul complex for $A_0 = \mathbb{k}$ is a finite free resolution.

Examples: Let R be the universal R-matrix of a quantum group U. Given a finite dimensional U-module V, define the permutation

$$P : V \otimes V \longrightarrow V \otimes V, \; v \otimes w \mapsto w \otimes v,$$

and let $\check{R} = PR$. Then $\check{R} \in \mathrm{End}_{\mathrm{U}}(V \otimes V)$.

The $\check{R}$-matrix has a characteristic polynomial of the form

$$\prod_{i=1}^{k_+} \left(x - q^{\chi_i^{(+)}}\right) \prod_{j=1}^{k_-} \left(x + q^{\chi_i^{(-)}}\right)$$

for some positive integers $k_\pm$, where $\chi_i^{(+)}$ and $\chi_i^{(-)}$ are integers related to eigenvalues of Casimir operators. Let

$$I_- = \prod_{i=1}^{k_+} \left(\check{R} - q^{\chi_i^{(+)}}\right) (V \otimes V), \tag{1}$$

which is a U-submodule of $V \otimes V$. We set

$$A = \Bbbk\{V, I_-\}.$$

Write $R = \sum_t \alpha_t \otimes \beta_t$.

Lemma

Let A, B and C be locally finite U-module algebras. Then

1. *$A \otimes_{\Bbbk} B$ forms a locally finite U-module algebra with the multiplication defined for all $a \otimes b, a' \otimes b' \in A \otimes B$ by*

$$(a \otimes b)(a' \otimes b') = \sum_t a(\beta_t \cdot a') \otimes (\alpha_t \cdot b)b'.$$

2. *$(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ are canonically isomorphic as U-module algebras.*

Thus given a finite dimensional U-module V, we have a U-module algebra

$$\Bbbk\{V, I_-\}^{\otimes m} \quad \text{for each positive integer } m.$$

Theorem

Let V be the natural $\mathrm{U}_q(\mathfrak{g})$-module for $\mathfrak{g} = sl_n$, so_n or sp_{2n}. Then $S_q(V^m) := \Bbbk\{V, I_-\}^{\otimes m}$ is a quantum symmetric algebra for all m. Furthermore, $S_q(V^m)$ is Noetherian.

The case $\mathfrak{g} = sl_n$ is familiar. The corresponding $S_q(V^m)$ is generated by x_{ij} $(1 \leq i \leq m,\ 1 \leq j \leq n)$ subject to the following relations

$$\begin{aligned}
x_{ij}x_{ik} &= q^{-1}x_{ik}x_{ij}, && j < k,\\
x_{ij}x_{kj} &= q^{-1}x_{kj}x_{ij}, && i < k,\\
x_{ij}x_{kl} &= x_{kl}x_{ij}, && i < k, j > l,\\
x_{ij}x_{kl} &= x_{kl}x_{ij} - (q - q^{-1})x_{il}x_{kj}, && i < k, j < l.
\end{aligned} \tag{2}$$

This is the coordinate algebra of a quantum $m \times n$ matrix.

For all the $S_q(V^m)$ described in the Theorem,

$$K_i^{\mathrm{U}_q(\mathfrak{g})}(S_q(V^m)) \cong K_i(\mathrm{U}_q(\mathfrak{g})\text{-}\mathbf{mod}), \quad \text{for all } i.$$

In particular, $K_0(\mathrm{U}_q(\mathfrak{g})\text{-}\mathbf{mod})$ is the Grothendieck group of $\mathrm{U}_q(\mathfrak{g})$-**mod**.

The usual algebraic K-groups of $S_q(V^m)$ are given by

$$K_i(S_q(V^m)) = K_i(\mathbb{k}), \quad \text{for all } i.$$

Quantum homogenous spaces

Present the quantum group U over $\Bbbk = \mathbb{C}(q)$ in terms of the usual generators $\{e_i, f_i, k_i^{\pm 1} | i = 1, 2, \ldots, r\}$ and standard relations.

Matrix elements of the U-representations associated with objects of U-**mod** span a Hopf subalgebra $A_{\mathfrak{g}}$ of the finite dual of U.

There exist two actions R and L of U on $A_{\mathfrak{g}}$ defined by

$$R_x f = \sum_{(f)} f_{(1)} < f_{(2)}, x >, \qquad L_x f = \sum_{(f)} < f_{(1)}, S(x) > f_{(2)}$$

for all $x \in \mathrm{U}$ and $f \in A_{\mathfrak{g}}$. The actions commute.
$A_{\mathfrak{g}}$ forms a U-module algebra under either R or L.

Let $\Theta \subset \{1, 2, \ldots, r\}$. Denote by $\mathrm{U}_q(\mathfrak{l})$ the Hopf subalgebra of U generated by the elements of $\{k_i^{\pm} \mid 1 \leq i \leq r\} \cup \{e_j, f_j \mid j \in \Theta\}$. Define

$$A = \{f \in A_{\mathfrak{g}} \mid L_x(f) = \epsilon(x) f, \ \forall x \in \mathrm{U}_q(\mathfrak{l})\}. \tag{3}$$

Lemma

The subspace A forms a U-module algebra under the action R. Furthermore, A is (both left and right) Noetherian.

The algebra A is the quantum analogue of the algebra of functions on G/K for a compact connected Lie group G and a closed subgroup K with $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{L}ie(G)$ and $\mathfrak{l} = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{L}ie(K)$.

Call the noncommutative space determined by the algebra a quantum homogeneous space.

Theorem

There is an abelian group isomorphism

$$K_i^{\mathrm{U}}(A) \cong K_i(\mathrm{U}_q(\mathfrak{l})\text{-}\mathbf{mod}) \quad \textit{for each } i \geq 0,$$

*where $\mathrm{U}_q(\mathfrak{l})$-**mod** is the category of finite dimensional semi-simple left $\mathrm{U}_q(\mathfrak{l})$-modules.*

For any object Ξ in $\mathrm{U}_q(\mathfrak{l})$-**mod**, define

$$\mathcal{S}(\Xi) = \left\{ \zeta \in \Xi \otimes A_{\mathfrak{g}} \,\middle|\, \sum_{(x)} (x_{(1)} \otimes L_{x_{(2)}})\zeta = \epsilon(x)\zeta, \ \forall x \in \mathrm{U}_q(\mathfrak{l}) \right\}. \quad (4)$$

Then $\mathcal{S}(\Xi)$ belongs to $\mathcal{M}(A, \mathrm{U})$ with A- and U-actions defined for any $b \in A$, $x \in \mathrm{U}$ and $\zeta = \sum v_i \otimes a_i \in \mathcal{S}(\Xi)$ by

$$\begin{aligned} b\zeta &= \sum v_i \otimes ba_i, \\ x\zeta &= (\mathrm{id}_\Xi \otimes R_x)\zeta = \sum v_i \otimes R_x(a_i). \end{aligned}$$

Theorem

1. *Let V be the restriction of a finite dimensional U-module to a $\mathrm{U}_q(\mathfrak{l})$-module. Then $\mathcal{S}(V) \cong V \otimes_{\mathbb{k}} A$ in $\mathcal{M}(A, \mathrm{U})$.*
2. *For any object Ξ in $\mathrm{U}_q(\mathfrak{l})$-**mod**, $\mathcal{S}(\Xi)$ belongs to $\mathcal{P}(A, \mathrm{U})$.*

Extend (4) to a covariant functor

$$\mathcal{S} : \mathrm{U}_q(\mathfrak{l})\text{-}\mathbf{mod} \longrightarrow \mathcal{P}(A, \mathrm{U}), \tag{5}$$

which applies to objects of $\mathrm{U}_q(\mathfrak{l})$-**mod** according to (4) and sends a morphism f to $f \otimes \mathrm{id}_{A_\mathfrak{g}}$.

Let $I = \{f \in A | f(1) = 0\}$, which is an ideal of A and forms a $\mathrm{U}_q(\mathfrak{l})$-module algebra under the restriction of the action R. Thus for any U-equivariant A-module M, IM is a $\mathrm{U}_q(\mathfrak{l})$-equviaraint A-submodule of M.

For M in $\mathcal{M}(A, \mathrm{U})$, let $M_0 = M/IM$. We now have a covariant functor

$$\mathcal{E} : \mathcal{M}(A, \mathrm{U}) \longrightarrow \mathrm{U}_q(\mathfrak{l})\text{-}\mathbf{mod},$$

which sends an object M in $\mathcal{M}(A, \mathrm{U})$ to M_0, and is defined for morphisms in the obvious way.

We restrict the functor to the full subcategory $\mathcal{P}(A, \mathrm{U})$ to obtain a covariant functor

$$\mathcal{E}_{\mathcal{P}} : \mathcal{P}(A, \mathrm{U}) \longrightarrow \mathrm{U}_q(\mathfrak{l})\text{-}\mathbf{mod}. \tag{6}$$

Theorem

There are natural isomorphisms $\mathcal{S} \circ \mathcal{E}_{\mathcal{P}} \cong \mathrm{id}_{\mathcal{P}(A,\mathrm{U})}$ *and* $\mathcal{E}_{\mathcal{P}} \circ \mathcal{S} \cong \mathrm{id}_{\mathrm{U}_q(\mathfrak{l})-\mathbf{mod}}$, *thus the categories* $\mathrm{U}_q(\mathfrak{l})$-**mod** *and* $\mathcal{P}(A, \mathrm{U})$ *are equivalent.*

The theorem on the equivariant K-groups of the quantum homogeneous spaces immediately follows from this result.

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